Reasoning under severe uncertainty: exercices

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1 Subjective probabilities

Exercice 1 (Irrational bets) You bet on raining ($x = R$) or not raining ($x = \neg R$). You specify the following prices :

$$P(R) = 0.5, P(\neg R) = 0.7$$

1. Show that you are not rational by exhibiting a dutch book (combination of bets leading to a sure loss for you)
2. Propose a possible repair of the bets, modifying only one of them.

Exercice 2 (Linearity) Show that if your bets are rational, then for a gamble f, a positive number α and a value $\mu \in \mathbb{R}$, we have

$$P(\alpha f + \mu) = \alpha P(f) + \mu$$

2 Lower previsions and equivalent notions

Exercice 1 (Coherence for lower previsions) Given a space $\mathcal{X} = \{x_1, x_2, x_3\}$, I define the following lower prevision :

$$P(f) = \frac{1}{2} \inf_x f(x) + \frac{1}{2} \max_x f(x)$$

1. Is it coherent ?

Exercice 2 (Coherence for desirable gambles) Given a space $\mathcal{X} = \{x_1, x_2, x_3\}$, are the following sets of desirable gambles coherent ?

1. $\mathcal{D}_1 = \{f(x_1) + f(x_2) + f(x_3) \geq 0\}$
2. $\mathcal{D}_2 = \{\max\{f(x_1), f(x_2), f(x_3)\} \geq 0\}$

Exercice 3 (Playing with credal sets) As a football match is coming, you plan to bet on whether the first team will be Winning, Tying or Loosing to the second. Your opinion on that on the space $\mathcal{X} = \{W, T, L\}$ is given by the following upper and lower probability bounds on atoms :

$$\underline{P}(\{W\}) = 0.1 \quad \overline{P}(\{W\}) = 0.3$$
$$\underline{P}(\{T\}) = 0.2 \quad \overline{P}(\{T\}) = 0.4$$
$$\underline{P}(\{L\}) = 0.3 \quad \overline{P}(\{L\}) = 0.5$$

1. Draw the associated probability set, if any
2. What are the extreme points of this probability set (none if it is empty)
3. Compute the lower probability $\underline{P}(A)$ for all events
4. Assume someone proposes to you the following bet : 10 if the team wins, 0 if it's a tie-break, and -5 if the team loses. According to your beliefs, at what entry price you would accept this bet ?

3 Practical models of uncertainty

Exercice 1 (2-monotone or not ?) Consider the knowledge given in Exercice 2.3 concerning the football match. Answer the following questions :

1. Is the obtained probability 2-monotone ?
2. Is the obtained probability completely monotone, or in other words, is it a belief function ?
3. Depending on your answer and of what you used to answer Question 2.3.4., can you propose/apply another scheme ?
4. Depending on your answer, either check that using a permutation indeed gives you an extreme point, or find a permutation that does not
5. Generally speaking, can you link this previous technique to the fact that Choquet integral gives you the lower expected bound ?

Exercice 2 (Manipulating belief functions) The pokemon league will soon finish, and five participants will fight for the final victory :

$$(a)rticuno,\ (b)ulbasaur,\ (c)harizard,\ (d)ragonite,\ (e)lectrode.$$

A survey was performed among 100 experts to know their opinion about the possible winner. The results were as follows :

— 30 experts think the winner will be in $\{a, c\}$
— 25 experts think the winner will be in $\{b, c, d\}$
— 15 experts think the winner will be $\{a\}$
— 5 experts think the winner will be in $\{c, d, e\}$
— 20 experts think the winner will be in $\{b, c\}$
— 5 experts admit having no knowledge in those particular pokemons, and say anyone in $\{a, b, c, d, e\}$ could win.

1. Could you model the information as a belief function ? If yes, how ?
2. Is the obtained upper probability a possibility measure ? If yes, prove it ; if no, provide two events falsifying the properties of possibility measures.
3. Assume someone offer you the following bet :
   — You win 100 Pokémon Dollars if a wins,
   — You lose 30 Pokémon Dollars if c wins,
   — You win 500 Pokémon Dollars if e wins.
   Assuming that you trust the survey result to form your opinion on the future result, is this bet desirable to you ? To which price would you be inclined to buy or sell it ? Can you give two ways to compute those ?

4 Deciding with probability sets

Exercice 1 (Applying different decision rules) We consider a coin tossing problem where $\mathcal{X} = \{H, T\}$, and all we know is that $p(H) \in [0.28, 0.7]$. We also consider the following actions :

a_i	H	T
a_1	4	0
a_2	0	4
a_3	3	2
a_4	1/2	3
a_5	47/20	47/20
a_6	41/10	−3/10

1. What can we say about $p(T)$?
2. Can you provide a graphical representation of the expected payoffs for $p(H) \in [0.28, 0.7]$?
3. Compute the decision sets $Opt_{\succ_i}(\mathcal{M}, \mathcal{A})$ for $\succeq_{MM}, \succeq_{Mm}, \succ_{ID}, \succ_L, \succ_D$ as well as $Opt_E(\mathcal{M}, \mathcal{A})$.

5 Learning with IP

Exercice 1 (Cats, dogs, and others) Given the space $\mathcal{Y} = \{(D)ogs, (C)ats, (H)orse, (R)abbit\}$, the features $(E)ars, (T)ails, (H)air$ each having for modalities $\mathcal{X}_i = \{(L)ong, (M)edium, (S)hort\}$ and the following number of observations :

	D	C	H	R
n_i	25	30	20	25

Table 1 – Prior observations of classes

Ears

n_i/Y	D	C	H	R
L	9	0	2	16
M	8	18	15	7
S	8	12	3	2

Tails

n_i/Y	D	C	H	R
L	15	10	15	0
M	6	20	5	8
S	4	0	0	17

Hairs

n_i/Y	D	C	H	R
L	11	15	5	0
M	7	5	2	5
S	7	10	13	20

Table 2 – Prior observations of classes

1. Assuming $s = 2$ and an imprecise Dirichlet model, can you give the (conditional) probability sets generated by those data ?
2. Assuming we use a NCC classifier, and assuming you observe an animal with Long Ears, Medium tails and Long hair. What are the probability bounds of it being a cat ?
3. Still assuming an NCC model, if K is the number of categories of the distributions, assume you contract each interval $[\underline{p}, \overline{p}]$ into $[(1-\epsilon)\underline{p} + \epsilon(1/K), (1-\epsilon)\underline{p} + \epsilon(1/K)]$. If $\epsilon = 0.05$, what happens to the previous probability bounds ?
4. Still assuming an NCC model, using the same contracted graphs, compute the whole dominance graphs of the classes in the case where you observe an animal with Medium Ears, Long tails and Short hair ?