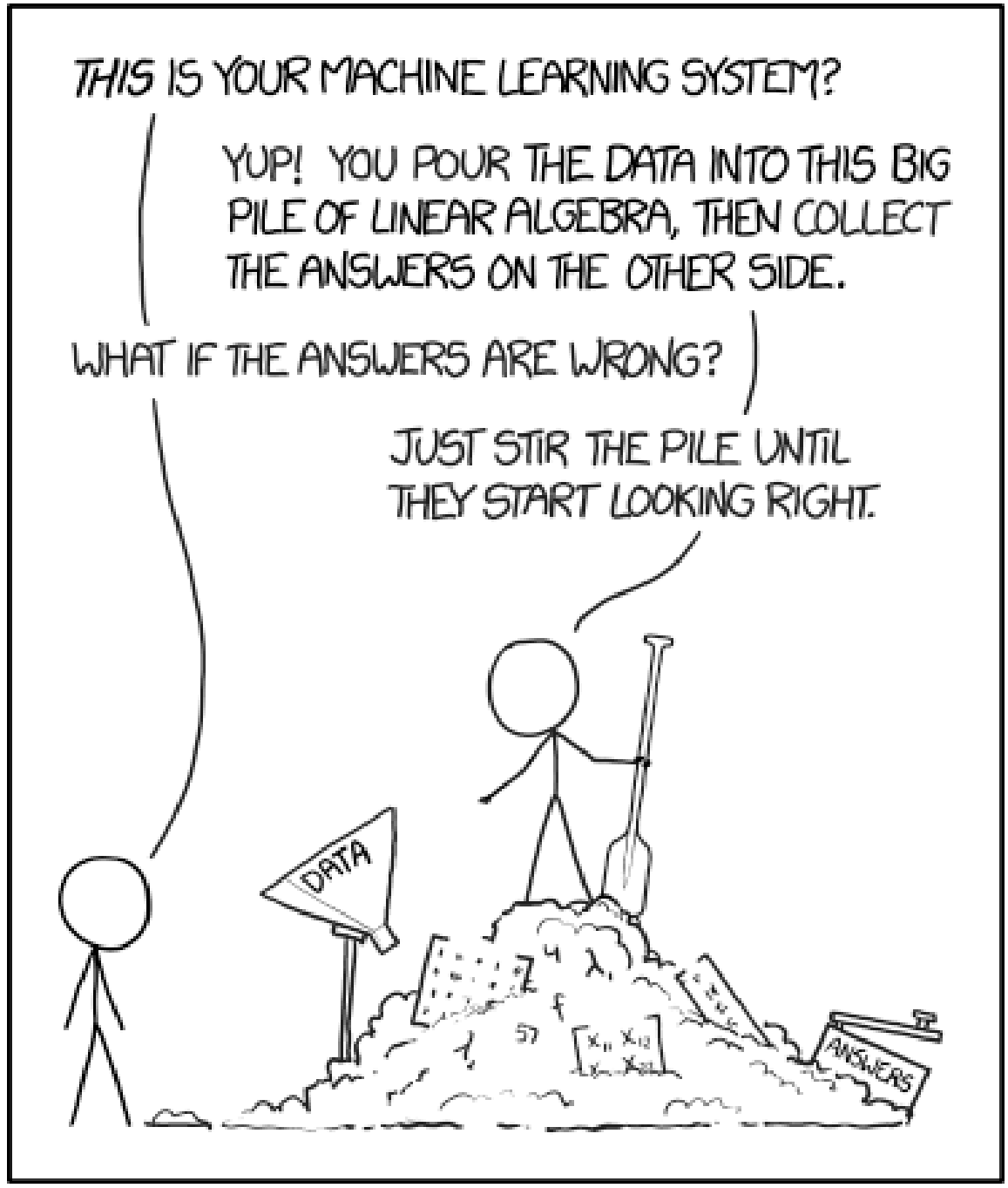

THIS IS YOUR MACHINE LEARNING SYSTEM?
YUP! YOU POUR THE DATA INTO THIS BIG PILE OF LINEAR ALGEBRA, THEN COLLECT THE ANSWERS ON THE OTHER SIDE.
WHAT IF THE ANSWERS ARE WRONG?
JUST STIR THE PILE UNTIL THEY START LOOKING RIGHT.
DATA
ANSWERS

# Uncertainty reasoning and machine learning

## Some first probabilistic and credal classifiers

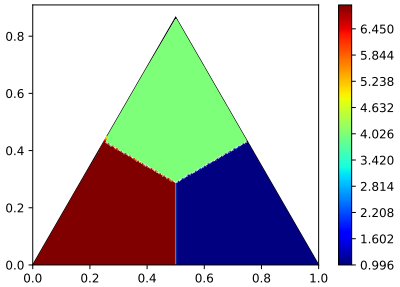
**Vu-Linh Nguyen**

**Chaire de Professeur Junior, Laboratoire Heudiasyc**
**Université de technologie de Compiègne**

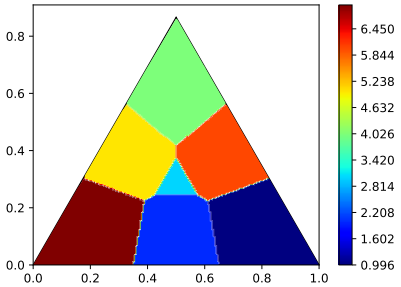
**AOS4 master courses**

# Optimal Decision Rules

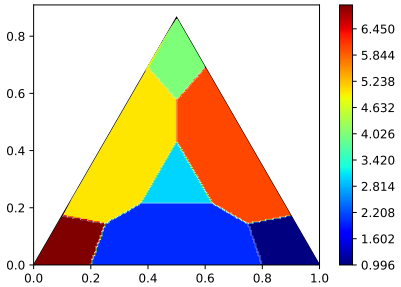
## Probabilistic approach




0/1 loss




$u_{1.6}$




$u_{2.2}$

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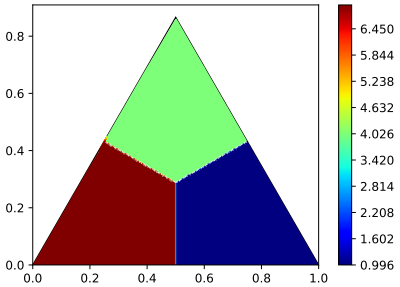


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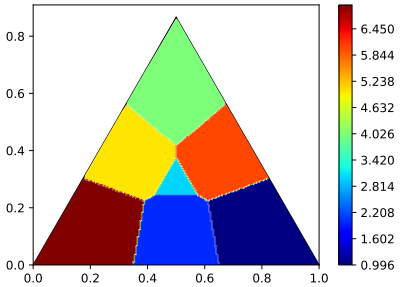


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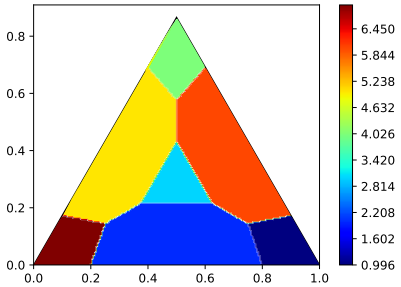
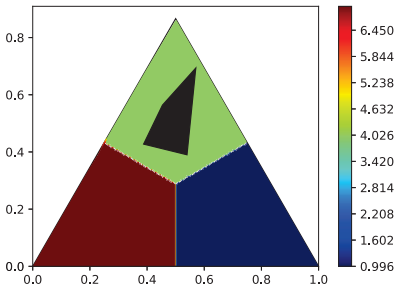
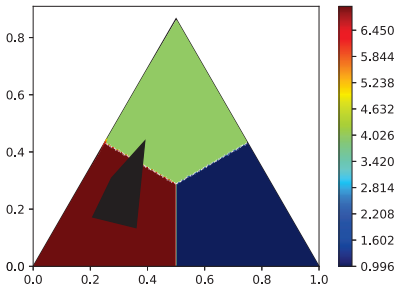
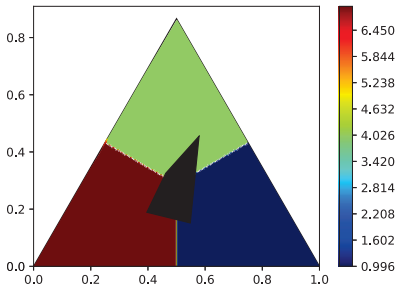



$u_{2.2}$

## Credal approach



# Computational Aspects

## Basic setup and assumption

- Given training data $\boldsymbol{D} \subset \mathscr{X} \times \mathscr{Y}$
- $\boldsymbol{D}$ is used to estimate a classifier, which predicts, for each $\boldsymbol{x}$, $P(Y|\boldsymbol{x})$

## Optimal decision rules

- The Bayes-optimal prediction of any $\ell : \mathscr{Y} \times \mathscr{Y} \longmapsto \mathbb{R}_+$ on $\boldsymbol{x}$ is
$$y_\ell^P \in \operatorname*{argmin}_{\overline{y} \in \mathscr{Y}} \sum_{y \in \mathscr{Y}} \ell(\overline{y}, y) P(y|\boldsymbol{x})$$
- The Bayes-optimal prediction of any $\mathscr{L} : 2^{\mathscr{Y}} \setminus \{\emptyset\} \times \mathscr{Y} \longmapsto \mathbb{R}_+$ on $\boldsymbol{x}$ is
$$S_{\mathscr{L}}^P \in \operatorname*{argmin}_{\overline{S} \subset \mathscr{Y}} \sum_{y \in \mathscr{Y}} \mathscr{L}(\overline{S}, y) P(y|\boldsymbol{x})$$

# Computational Aspects (Cont.)

**Basic setup and assumption**

- Given training data $\boldsymbol{D} \subset \mathscr{X} \times \mathscr{Y}$
- $\boldsymbol{D}$ is used to estimate a classifier, which predicts, for each $\boldsymbol{x}$, $\mathscr{P}(Y|\boldsymbol{x})$

**E-admissibility Rule [11, 13]**:

- Let $\ell : \mathscr{Y} \times \mathscr{Y} \longmapsto \mathbb{R}_+$ be a loss. An optimal prediction is

$$S^E_{\ell,\mathscr{P}} = \{y \in \mathscr{Y} | \exists P(Y|\boldsymbol{x}) \in \mathscr{P}(Y|\boldsymbol{x}) \text{ s.t. } y = y^P_\ell\}.$$

- Computation: Solving linear programs, etc.

# Beyond Multi-Class Classification

**Other predictive tasks**:

- Multi-Label Classification
- Multi-Dimensional Classification
- Multi-Target Prediction

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**Other predictive tasks**:

- Multi-Label Classification
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**Practical Challenges**:

- Mixed features (e.g., Multimodal inputs)
- Insufficient training data: Imbalance, Scarce, Incomplete, Noise
- Incomplete test inputs

# Multi-label classification with partial abstention

- **Precise predictions**: $\mathscr{Y} = \{0,1\}^K$
- **Set-valued predictions**: $\mathscr{Y}_{\text{set}} = 2^{\mathscr{Y}}$
- **Predictions with partial abstention**: $\mathscr{Y}_{\text{par}} = \{0,1,\perp\}^K \subsetneq \mathscr{Y}_{\text{set}}$

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## Multilabel Classification with Partial Abstention: Bayes-Optimal Prediction under Label Independence

**Vu-Linh Nguyen** V.L.NGUYEN@TUE.NL
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*Eindhoven University of Technology, The Netherlands*

**Eyke Hüllermeier** EYKE@LMU.DE
*Department of Computer Science*
*University of Munich (LMU), Germany*

# Objectives

After this lecture, students should be able to

- use IDM and related models in Naïve credal classifier (NCC) [3]
- use IDM and related models in decision trees [9]

# Outline


- Graphical Interpretation of Probabilistic Models
- Naïve Bayesian/Credal classifiers
- Decision Trees
- Bayesian Neural Networks
- Summary and Outlook

# How to interpret a decision tree?

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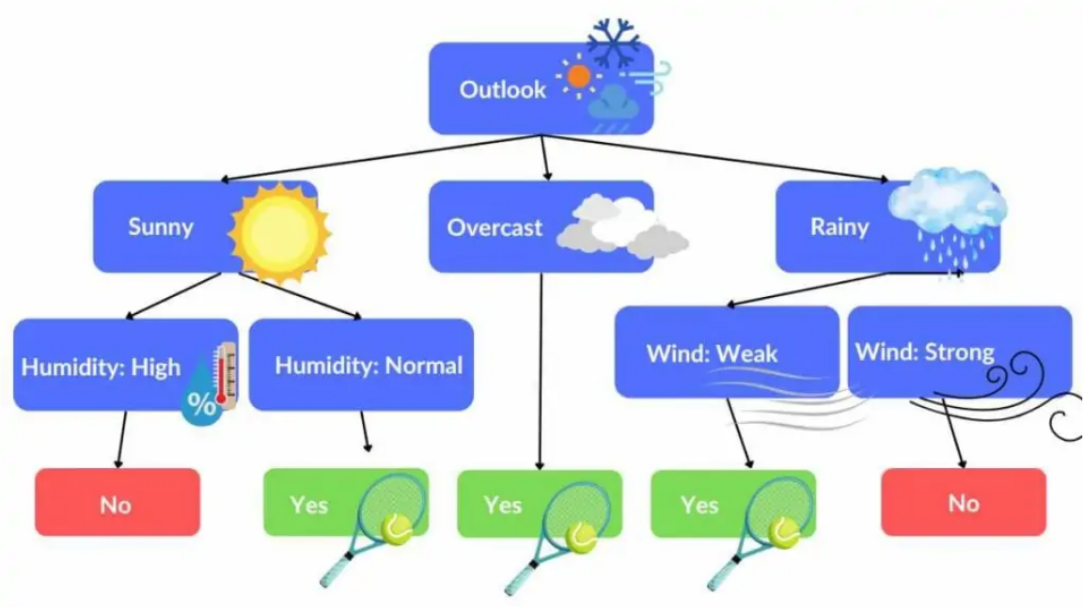


Source: `https://spotintelligence.com/2024/05/22/decision-trees-in-ml/`

# How to interpret a (feedforward) neural network?

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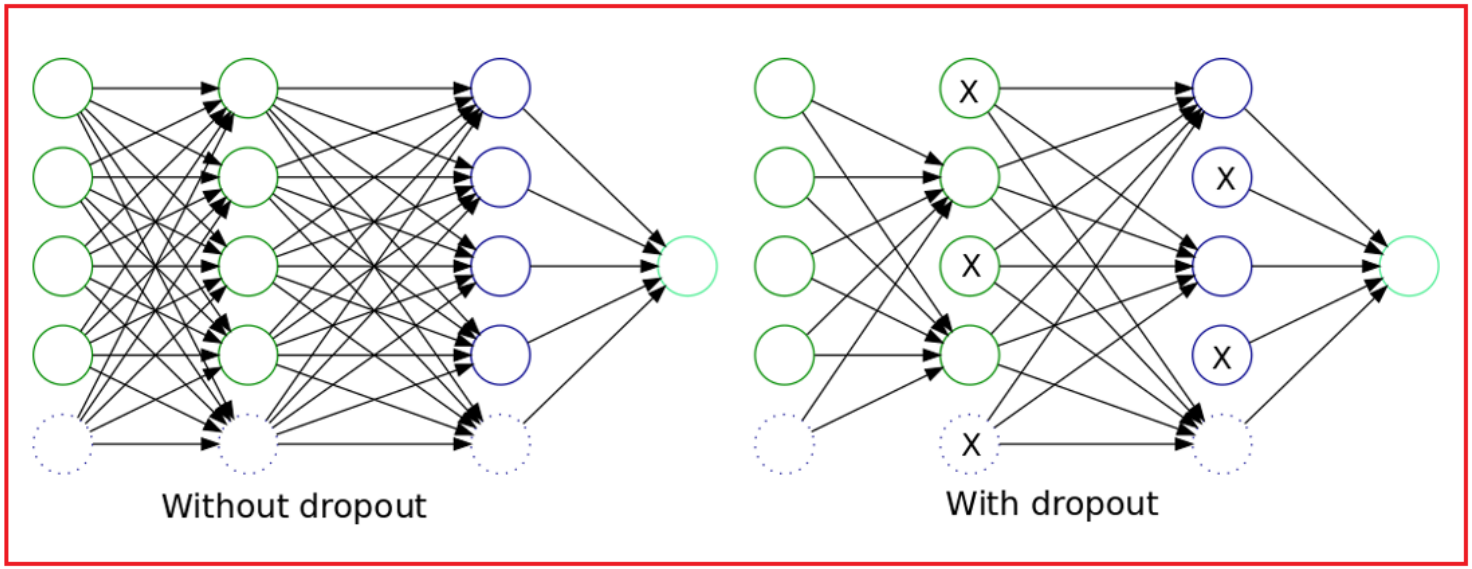

# Probabilistic Models: Graphical Interpretation [6, 10]

## Basic setup

- A set of features $\boldsymbol{X} = \{X^1, \ldots, X^M\}$; $[M] := \{1, \ldots, M\}$
- A class variable $Y$ whose outcome $y \in \mathscr{Y}$

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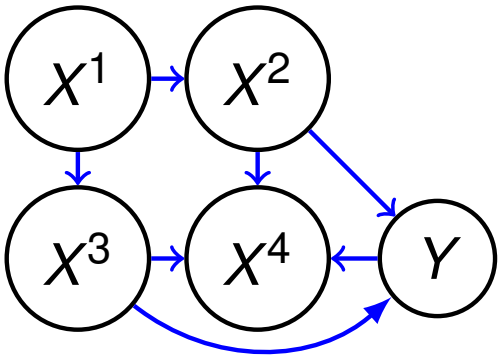


This DAG (model structure) tells us:

- $\text{pa}(Y) = \{X^2, X^3\}$, $\text{pa}(X^1) = \emptyset$
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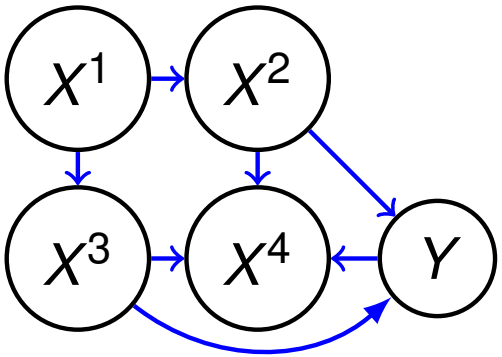


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## Probabilistic Models:

- Expressing $P(Y, \boldsymbol{X})$ using the **chain rule** (probability):

$$P(Y, \boldsymbol{X}) = P(Y|\text{pa}(Y)) \prod_{m=1}^{M} P(X^m|\text{pa}(X^m)).$$

# Probabilistic Models: Model Families [10]

**Probabilistic Models**:

- Estimate $P(Y, \boldsymbol{X})$
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**Extreme Cases**:

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**Model Families**:

- How to encode/parametrize $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$.
- How to estimate $P(Y, \boldsymbol{X})$ from training data.

# Credal (Imprecise Probability) Models [5]

## Basic setup

- A set of features $\boldsymbol{X} = \{X^1, \ldots, X^M\}$
- A class variable $Y$ whose outcome $y \in \mathscr{Y}$

## Credal Models:

- $\mathscr{P} := \{P(Y, \boldsymbol{X}) | P \text{ is compatible with knowledge/data}\}$
- Chain rule (probability):

$$P(Y, \boldsymbol{X}) = P(Y|\text{pa}(Y)) \prod_{m=1}^{M} P(X^m|\text{pa}(X^m)).$$

## Extreme Cases:

- Discriminative models: $Y \notin \text{pa}(X^m)$, $m \in [M] := \{1, \ldots, M\}$
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## Model Families:

- How to encode/parametrize $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$.
- How to estimate $\mathscr{P}(Y, \boldsymbol{X})$ from training data.

# Assumptions and Questions

**Assumption and desirable property**:

A1. $X^m$, $m \in [M] := \{1, \ldots, M\}$, are always made available

P1. Best estimates of $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$ can be found given (training) data.

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**Questions** (Exercise):

- Does the P1 hold for Naïve Bayes Classifier?
- Does the P1 hold for Decision trees?

**Questions** (which will not be discussed in this lecture):

- What may happen if $X^m$, $m \in [M]$, can be partially given?
- What may happen if best estimates of $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$ may not be found?

# The Next Slides

We shall elaborate on how to solve classification task using

- Naïve Bayesian classifier (NBC) (an example of generative model)
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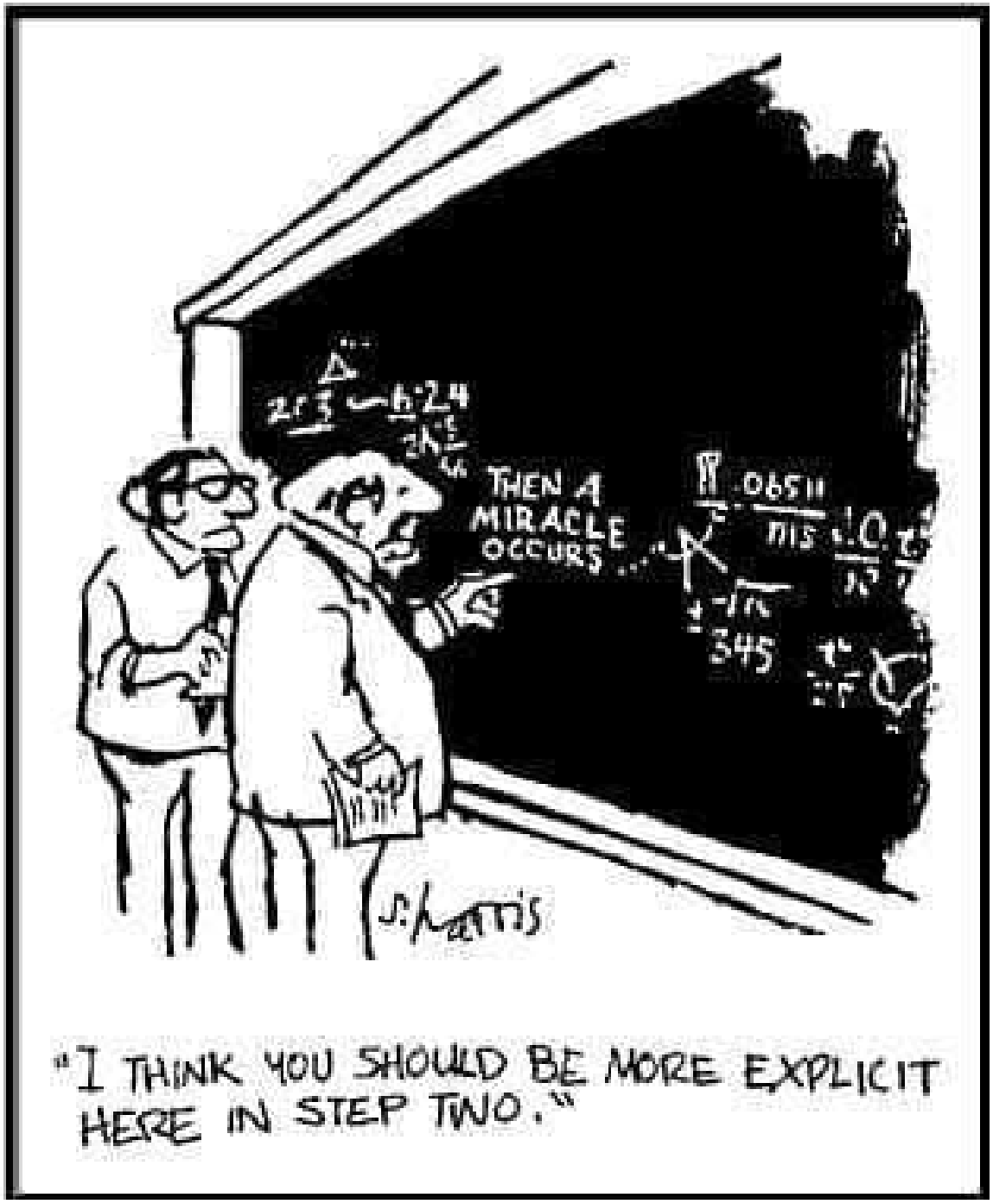
How IDM (Lecture 1) can be used to generalize NBC and DTs to

- cope with the case of small and partial/missing data
- make set-valued predictions under the presence of uncertainty

We would also discuss (if we have time) the cases of

- Ensembles (Trees, Neural Nets, etc.)
- Bayesian Neural Nets




"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

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# Generative Models

**Probabilistic Models**:

- Estimate $P(Y, \boldsymbol{X})$
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**Extreme Cases**:

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**Model Families**:

- How to encode/parametrize $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$.
- How to estimate $P(Y, \boldsymbol{X})$ from training data.

# Generative Models: Structure

Let's start with an example where one wishes to model

$$P(Y, \boldsymbol{X}) = P(Y, X^1, X^2, X^3, X^4).$$

Chain rule (probability):

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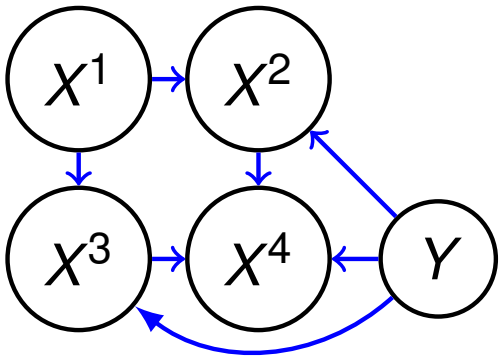


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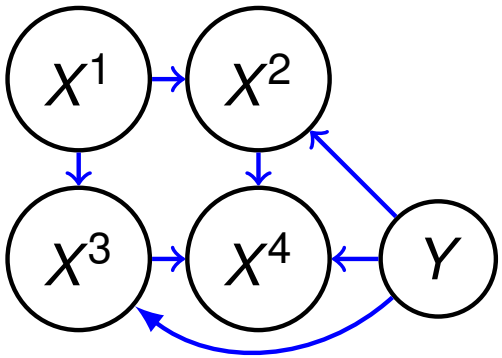


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The chain rule gives us

$$P(Y, \boldsymbol{X}) = P(Y)P(X^1)P(X^2|Y, X^1)P(X^3|Y, X^1)P(X^4|Y, X^2, X^3).$$

# Naïve Bayesian classifier (NBC)

**Comments**:

- NBC is a generative model with no arc $X' \longrightarrow X$
- Chain rule gives us

$$P(Y, \boldsymbol{X}) = P(Y) \prod_{m=1}^{M} P(X^m | Y).$$

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- Chain rule gives us

$$P(Y, \boldsymbol{X}) = P(Y) \prod_{m=1}^{M} P(X^m | Y).$$

To solve the **classification task**,

- joint probability distribution $P(Y, \boldsymbol{X})$ is learn from training data $\boldsymbol{D}$
- conditional distribution $P(Y|\boldsymbol{X})$ is extracted using **Bayes' theorem**

$$P(y|\boldsymbol{x}) = \frac{P(y, \boldsymbol{x})}{\sum_{y' \in \mathcal{Y}} P(y', \boldsymbol{x})} = \frac{P(y) \prod_{m=1}^{M} P(x^m|y)}{\sum_{y' \in \mathcal{Y}} P(y') \prod_{m=1}^{M} P(x^m|y')}. \quad (1)$$

# Estimate Parameters of NBC

**Basic setup**:

- A class variable $Y$ with $K$ possible values: $\mathscr{Y} = \{y^1, \dots y^K\}$
- $M$ discrete features: $\boldsymbol{X} = (X^1, \dots, X^M)$
- Feature $X^m$ has $Q_m$ possible values: $\mathscr{X}^m = \{x^{m,1}, \dots x^{m,Q_m}\}$

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**Task**: Finding the best estimate of

- $P_k := P(y^k)$, $k \in [K]$
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**Probability axioms**:

- $\sum_{k=1}^{K} P_k = 1$
- $\sum_{q_m=1}^{Q_m} P_k^{m,q_m} = 1$ when fixing $k$ and $m$

# Maximum Likelihood Estimate

**Basic setup**: Given training data $\boldsymbol{D} = \{(y_1, \boldsymbol{x}_1), \ldots, (y_N, \boldsymbol{x}_N)\}$, count

- $n_k$: Number of training instances with label $y^k$
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MLE gives us the best estimates

$$P_k := n_k / N \tag{2}$$

$$P_k^{m,q_m} := n^{m,q_m} / n_k \tag{3}$$

# MLE with Examples

- $\mathscr{Y} = \{A, B, C\}$
- $\mathscr{X}^1 = \{d, e\}$
- $\mathscr{X}^2 = \{f, g, h\}$

| $n$ | $Y$ | $X^1$ | $X^2$ |
|---|---|---|---|
| 1 | A | d | f |
| 2 | A | d | g |
| 3 | A | e | g |
| 4 | B | d | f |
| 5 | B | e | g |
| 6 | C | d | f |
| 7 | C | e | f |
| 8 | C | e | g |

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| 4 | B | d | f |
| 5 | B | e | g |
| 6 | C | d | f |
| 7 | C | e | f |
| 8 | C | e | g |

$$n_A = 3 \quad n_B = 2 \quad n_C = 3$$
$$P_A = 3/8 \quad P_B = 1/4 \quad P_C = 3/8$$

| | | | |
|---|---|---|---|
| $n_A^{1,d} = 2$ | $n_A^{1,e} = 1$ | $P_A^{1,d} = 2/3$ | $P_A^{1,e} = 1/3$ |
| $n_B^{1,d} = 1$ | $n_B^{1,e} = 1$ | $P_B^{1,d} = 1/2$ | $P_B^{1,e} = 1/2$ |
| $n_C^{1,d} = 1$ | $n_C^{1,e} = 2$ | $P_C^{1,d} = 1/3$ | $P_C^{1,e} = 2/3$ |
| $n_A^{2,f} = 1$ | $n_A^{2,g} = 2$ | $P_A^{2,f} = 1/3$ | $P_A^{2,g} = 2/3$ |
| $n_B^{2,f} = 1$ | $n_B^{2,g} = 1$ | $P_B^{2,f} = 1/2$ | $P_B^{2,g} = 1/2$ |
| $n_C^{2,f} = 2$ | $n_C^{2,g} = 1$ | $P_C^{2,f} = 2/3$ | $P_C^{2,g} = 1/3$ |

$$n_A^{2,h} = 0 \quad n_B^{2,h} = 0 \quad n_C^{2,h} = 0$$
$$P_A^{2,h} = 0 \quad P_B^{2,h} = 0 \quad P_C^{2,h} = 0$$

# Conditional Probabilities

Given $\boldsymbol{x} = (x^{1,q_1}, \ldots, x^{M,q_M})$, for any $y^k \in \mathcal{Y}$:

$$P(y^k|\boldsymbol{x}) = \frac{P_k \prod_{m=1}^{M} P_k^{m,q_m}}{\sum_{y_{k'} \in \mathcal{Y}} P_{k'} \prod_{m=1}^{M} P_{k'}^{m,q_m}} \propto P'(y^k|\boldsymbol{x}) = P_k \prod_{m=1}^{M} P_k^{m,q_m}. \qquad (4)$$

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$$P_A = 3/8 \quad P_B = 1/4 \quad P_C = 3/8$$

| | | | |
|---|---|---|---|
| $P_A^{1,d} = 2/3$ | $P_A^{1,e} = 1/3$ | $P_A^{2,f} = 1/3$ | $P_A^{2,g} = 2/3$ |
| $P_B^{1,d} = 1/2$ | $P_B^{1,e} = 1/2$ | $P_B^{2,f} = 1/2$ | $P_B^{2,g} = 1/2$ |
| $P_C^{1,d} = 1/3$ | $P_C^{1,e} = 2/3$ | $P_C^{2,f} = 2/3$ | $P_C^{2,g} = 1/3$ |

$$P_A^{2,h} = 0 \quad P_B^{2,h} = 0 \quad P_C^{2,h} = 0$$

| $\boldsymbol{x}$ | $P'(A\|\boldsymbol{x})$ | $P'(B\|\boldsymbol{x})$ | $P'(C\|\boldsymbol{x})$ |
|---|---|---|---|
| $(d,f)$ | $1/12$ | $1/16$ | $1/12$ |
| $(e,h)$ | 0 | 0 | 0 |

# Optimal Decision Rules

If $\ell(y^{k'}, y^k) = \mathbb{1}(y^{k'} \neq y^k)$, then (Check!)

$$y_\ell^P(\boldsymbol{x}) = \operatorname*{argmax}_{y^k \in \mathcal{Y}} P'(y^k | \boldsymbol{x})$$

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|---|---|---|---|
| $(d,f)$ | $1/12$ | $1/16$ | $1/12$ |
| $(e,h)$ | 0 | 0 | 0 |

- If $\boldsymbol{x} = (d,f)$, then

$$y_\ell^P(\boldsymbol{x}) = \text{ either } A \text{ or } C, \tag{5}$$

- If $\boldsymbol{x} = (e,h)$, then

$$y_\ell^P(\boldsymbol{x}) = \text{ not well-defined }, \tag{6}$$

# NBC + MLE: Comments

- May lead to **indecision** and **not well-defined** $P(y|\boldsymbol{x})$

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|---|---|---|---|
| $(d,f)$ | $1/12$ | $1/16$ | $1/12$ |
| $(e,h)$ | 0 | 0 | 0 |

- May suffer from small numbers of observations
  - $n_k$: Number of training instances with label $y^k$
  - $n_k^{m,q_m}$: Number of training instances with label $y^k$ and feature $X^m$ takes value $x^{m,q_m}$

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- Does not (naturally) take into account missing/partial data

# NBC + Dirichlet Model (DM)

**Basic setup**: Given training data $\boldsymbol{D} = \{(y_1, \boldsymbol{x}_1), \ldots, (y_N, \boldsymbol{x}_N)\}$, count

- $n_k$: Number of training instances with label $y^k$
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DM gives Bayesian estimates

$$P_k := (n_k + \alpha_k) \big/ (N + s) = (n_k + s f_k) \big/ (N + s) \tag{7}$$

$$P_k^{m,q_m} := (n_k^{m,q_m} + \alpha_k^{m,q_m}) \big/ (n_k + s) = (n_k^{m,q_m} + s f_k^{m,q_m}) \big/ (n_k + s) \tag{8}$$

# NBC + Dirichlet Model (DM)

**Basic setup**: Given training data $\boldsymbol{D} = \{(y_1, \boldsymbol{x}_1), \ldots, (y_N, \boldsymbol{x}_N)\}$, count

- $n_k$: Number of training instances with label $y^k$
- $n_k^{m,q_m}$: Number of instances with $Y = y^k$ and feature $X^m = x^{m,q_m}$

DM gives Bayesian estimates

$$P_k := (n_k + \alpha_k)/(N+s) = (n_k + sf_k)/(N+s) \tag{7}$$

$$P_k^{m,q_m} := (n_k^{m,q_m} + \alpha_k^{m,q_m})/(n_k+s) = (n_k^{m,q_m} + sf_k^{m,q_m})/(n_k+s) \tag{8}$$

| Advocators | $\alpha_k$ or $\alpha_k^{m,q_m}$ | $s$ |
|---|---|---|
| Haldane (1948) | 0 | 0 |
| Perks (1947) | $1/\lvert\mathcal{Y}\rvert$ | 1 |
| Jeffreys (1946, 1961) | $1/2$ | $\lvert\mathcal{Y}\rvert/2$ |
| Bayes-Laplace | 1 | $\lvert\mathcal{Y}\rvert$ |

# NBC + DM with Examples

$$P_k := (n_k+1/3)/(N+1), \qquad P_k^{m,q_m} := (n_k^{m,q_m}+1/|\mathscr{X}^m|)/(n_k+1).$$

# NBC + DM with Examples

$$P_k := (n_k+1/3)/(N+1), \qquad P_k^{m,q_m} := (n_k^{m,q_m}+1/|\mathscr{X}^m|)/(n_k+1).$$

- $\mathscr{Y} = \{A, B, C\}$
- $\mathscr{X}^1 = \{d, e\}$
- $\mathscr{X}^2 = \{f, g, h\}$

| $n$ | $Y$ | $X^1$ | $X^2$ |
|---|---|---|---|
| 1 | A | d | f |
| 2 | A | d | g |
| 3 | A | e | g |
| 4 | B | d | f |
| 5 | B | e | g |
| 6 | C | d | f |
| 7 | C | e | f |
| 8 | C | e | g |

$$n_A = 3 \quad n_B = 2 \quad n_C = 3$$
$$P_A = 10/27 \quad P_B = 7/27 \quad P_C = 10/27$$

| | | | |
|---|---|---|---|
| $n_A^{1,d} = 2$ | $n_A^{1,e} = 1$ | $P_A^{1,d} = 5/8$ | $P_A^{1,e} = 3/8$ |
| $n_B^{1,d} = 1$ | $n_B^{1,e} = 1$ | $P_B^{1,d} = 3/6$ | $P_B^{1,e} = 3/6$ |
| $n_C^{1,d} = 1$ | $n_C^{1,e} = 2$ | $P_C^{1,d} = 3/8$ | $P_C^{1,e} = 5/8$ |
| $n_A^{2,f} = 1$ | $n_A^{2,g} = 2$ | $P_A^{2,f} = 4/12$ | $P_A^{2,g} = 7/12$ |
| $n_B^{2,f} = 1$ | $n_B^{2,g} = 1$ | $P_B^{2,f} = 4/9$ | $P_B^{2,g} = 4/9$ |
| $n_C^{2,f} = 2$ | $n_C^{2,g} = 1$ | $P_C^{2,f} = 7/12$ | $P_C^{2,g} = 4/12$ |

$$n_A^{2,h} = 0 \quad n_B^{2,h} = 0 \quad n_C^{2,h} = 0$$
$$P_A^{2,h} = 1/12 \quad P_B^{2,h} = 1/9 \quad P_C^{2,h} = 1/12$$

# Conditional Probabilities

Given $\boldsymbol{x} = (x^{1,q_1}, \ldots, x^{M,q_M})$, for any $y^k \in \mathcal{Y}$:

$$P(y^k|\boldsymbol{x}) = \frac{P_k \prod_{m=1}^{M} P_k^{m,q_m}}{\sum_{y_{k'} \in \mathcal{Y}} P_{k'} \prod_{m=1}^{M} P_{k'}^{m,q_m}} \propto P'(y^k|\boldsymbol{x}) = P_k \prod_{m=1}^{M} P_k^{m,q_m}. \qquad (9)$$

$$P_A = {}^{10}/_{27} \quad P_B = {}^{7}/_{27} \quad P_C = {}^{10}/_{27}$$

| | | | |
|---|---|---|---|
| $P_A^{1,d} = {}^{5}/_{8}$ | $P_A^{1,e} = {}^{3}/_{8}$ | $P_A^{2,f} = {}^{4}/_{12}$ | $P_A^{2,g} = {}^{7}/_{12}$ |
| $P_B^{1,d} = {}^{3}/_{6}$ | $P_B^{1,e} = {}^{3}/_{6}$ | $P_B^{2,f} = {}^{4}/_{9}$ | $P_B^{2,g} = {}^{4}/_{9}$ |
| $P_C^{1,d} = {}^{3}/_{8}$ | $P_C^{1,e} = {}^{5}/_{8}$ | $P_C^{2,f} = {}^{7}/_{12}$ | $P_C^{2,g} = {}^{4}/_{12}$ |

$$P_A^{2,h} = {}^{1}/_{12} \quad P_B^{2,h} = {}^{1}/_{9} \quad P_C^{2,h} = {}^{1}/_{12}$$

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| | | | |
|---|---|---|---|
| $P_A^{1,d} = {}^{5}/_{8}$ | $P_A^{1,e} = {}^{3}/_{8}$ | $P_A^{2,f} = {}^{4}/_{12}$ | $P_A^{2,g} = {}^{7}/_{12}$ |
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| $P_C^{1,d} = {}^{3}/_{8}$ | $P_C^{1,e} = {}^{5}/_{8}$ | $P_C^{2,f} = {}^{7}/_{12}$ | $P_C^{2,g} = {}^{4}/_{12}$ |

$$P_A^{2,h} = {}^{1}/_{12} \quad P_B^{2,h} = {}^{1}/_{9} \quad P_C^{2,h} = {}^{1}/_{12}$$

| $\boldsymbol{x}$ | $P'(A\|\boldsymbol{x})$ | $P'(B\|\boldsymbol{x})$ | $P'(C\|\boldsymbol{x})$ | $y_\ell^P(\boldsymbol{x})$ |
|---|---|---|---|---|
| $(d,f)$ | $\frac{10}{27}\frac{5}{8}\frac{4}{12}$ | $\frac{7}{27}\frac{3}{6}\frac{4}{9}$ | $\frac{10}{27}\frac{3}{8}\frac{7}{12}$ | C |
| $(e,h)$ | $\frac{10}{27}\frac{3}{8}\frac{1}{12}$ | $\frac{7}{27}\frac{3}{6}\frac{1}{9}$ | $\frac{10}{27}\frac{5}{8}\frac{1}{12}$ | C |

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|---|---|---|---|---|
| $(d,f)$ | $\frac{10}{27}\ \frac{5}{8}\ \frac{4}{12}$ | $\frac{7}{27}\ \frac{3}{6}\ \frac{4}{9}$ | $\frac{10}{27}\ \frac{3}{8}\ \frac{7}{12}$ | C |
| $(e,h)$ | $\frac{10}{27}\ \frac{3}{8}\ \frac{1}{12}$ | $\frac{7}{27}\ \frac{3}{6}\ \frac{1}{9}$ | $\frac{10}{27}\ \frac{5}{8}\ \frac{1}{12}$ | C |

- May suffer from small numbers of observations
  - $n_k$: Number of training instances with label $y^k$
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|---|---|---|---|---|
| $(d,f)$ | $\frac{10}{27}\ \frac{5}{8}\ \frac{4}{12}$ | $\frac{7}{27}\ \frac{3}{6}\ \frac{4}{9}$ | $\frac{10}{27}\ \frac{3}{8}\ \frac{7}{12}$ | C |
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- May suffer from small numbers of observations
  - $n_k$: Number of training instances with label $y^k$
  - $n_k^{m,q_m}$: Number of training instances with label $y^k$ and feature $X^m$ takes value $x^{m,q_m}$
- Does not (naturally) take into account missing/partial data

# Outline


- Graphical Interpretation of Probabilistic Models
- Naïve Bayesian/Credal classifiers
  - Naïve Bayesian classifier
  - Naïve Credal classifiers
- Decision Trees
- Bayesian Neural Networks
- Summary and Outlook

# Naïve Credal classifiers (NCC)

**Basic setup**: Given training data $\boldsymbol{D} = \{(y_1, \boldsymbol{x}_1), \ldots, (y_N, \boldsymbol{x}_N)\}$, count

- $n_k$: Number of training instances with label $y^k$
- $n_k^{m,q_m}$: Number of instances with $Y = y^k$ and feature $X^m = x^{m,q_m}$

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- $n_k^{m,q_m}$: Number of instances with $Y = y^k$ and feature $X^m = x^{m,q_m}$

Imprecise Dirichlet model (IDM) gives

$$\underline{P}_k := n_k / (N+s) \quad (10) \qquad \overline{P}_k := (n_k+s) / (N+s) \quad (12)$$

$$\underline{P}_k^{m,q_m} := n_k^{m,qm} / (n_k+s) \quad (11) \qquad \overline{P}_k^{m,q_m} := (n_k^{m,qm}+s) / (n_k+s) \quad (13)$$

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$$\underline{P}_k := n_k/(N+s) \quad (10) \qquad \overline{P}_k := (n_k+s)/(N+s) \quad (12)$$

$$\underline{P}_k^{m,q_m} := n_k^{m,qm}/(n_k+s) \quad (11) \qquad \overline{P}_k^{m,q_m} := (n_k^{m,qm}+s)/(n_k+s) \quad (13)$$

IDM + $\epsilon$ regularization [2]

$$\underline{P}_k := (n_k+s\underline{\epsilon}_k)/(N+s) \quad (14) \qquad \overline{P}_k := (n_k+s\overline{\epsilon}_k)/(N+s) \quad (16)$$

$$\underline{P}_k^{m,q_m} := (n_k^{m,qm}+s\underline{\epsilon}_k^{m,qm})/(n_k+s) \quad (15) \qquad \overline{P}_k^{m,q_m} := (n_k^{m,qm}+s\overline{\epsilon}_k^{m,qm})/(n_k+s) \quad (17)$$

# Interval Conditional Probabilities

Given a query instance $\boldsymbol{x} = (x^{q_1,1}, x^{q_2,2}, \ldots, x^{q_M,M})$, we have

$$1/\overline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\underline{\epsilon}_k}{n_k + s\overline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}} \right),$$

$$1/\underline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\overline{\epsilon}_k}{n_k + s\underline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}} \right).$$

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$$1/\underline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\overline{\epsilon}_k}{n_k + s\underline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}} \right).$$

$$\mathscr{P}(Y|\boldsymbol{x}) := \left\{ P(Y|\boldsymbol{x}) | P(y^k|\boldsymbol{x}) \in [\underline{P}(y^k|\boldsymbol{x}), \overline{P}(y^k|\boldsymbol{x})], \textstyle\sum_{k=1}^{K} P(y^k|\boldsymbol{x}) = 1 \right\}.$$

# Interval Conditional Probabilities

$$1/\overline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\underline{\epsilon}_k}{n_k + s\overline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}} \right),$$

$$1/\underline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\overline{\epsilon}_k}{n_k + s\underline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}} \right).$$

# Interval Conditional Probabilities

$$1/\overline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\underline{\epsilon}_k}{n_k + s\overline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}} \right),$$

$$1/\underline{P}(y^k|\boldsymbol{x}) - 1 = \sum_{k' \neq k} \left( \frac{n_{k'} + s\overline{\epsilon}_k}{n_k + s\underline{\epsilon}_k} \left( \frac{n_k + s}{n_{k'} + s} \right)^M \prod_{m=1}^{M} \frac{n_{k'}^{q_m,m} + s\overline{\epsilon}_k^{m,q_m}}{n_k^{q_m,m} + s\underline{\epsilon}_k^{m,q_m}} \right).$$

$s = 1 \quad \underline{\epsilon}_k = 0.01 \quad \overline{\epsilon}_k = 0.99 \quad n_A = 3 \quad n_B = 2 \quad n_C = 3$

| | | | |
|---|---|---|---|
| $n_A^{1,d} = 2$ | $n_A^{1,e} = 1$ | $n_A^{2,f} = 1$ | $n_A^{2,g} = 2$ |
| $n_B^{1,d} = 1$ | $n_B^{1,e} = 1$ | $n_B^{2,f} = 1$ | $n_B^{2,g} = 1$ |
| $n_C^{1,d} = 1$ | $n_C^{1,e} = 2$ | $n_C^{2,f} = 2$ | $n_C^{2,g} = 1$ |

$$n_A^{2,h} = 0 \quad n_B^{2,h} = 0 \quad n_C^{2,h} = 0$$

| $\boldsymbol{x}$ | $\underline{P}(A\|\boldsymbol{x})$ | $\underline{P}(B\|\boldsymbol{x})$ | $\underline{P}(C\|\boldsymbol{x})$ | $\overline{P}(A\|\boldsymbol{x})$ | $\overline{P}(B\|\boldsymbol{x})$ | $\overline{P}(C\|\boldsymbol{x})$ |
|---|---|---|---|---|---|---|
| $(d,f)$ | ??? | ??? | ??? | ??? | ??? | ??? |
| $(e,h)$ | ??? | ??? | ??? | ??? | ??? | ??? |

# Making Set-Valued Predictions (Recap)

## Basic setup and assumption

- Given training data $\boldsymbol{D} \subset \mathscr{X} \times \mathscr{Y}$
- $\boldsymbol{D}$ is used to estimate a classifier, which predicts, for each $\boldsymbol{x}$, $\mathscr{P}(Y|\boldsymbol{x})$

## E-admissibility Rule [11, 13]:

- Let $\ell : \mathscr{Y} \times \mathscr{Y} \longmapsto \mathbb{R}_+$ be a loss. An optimal prediction is

$$S^E_{\ell,\mathscr{P}} = \{y \in \mathscr{Y} | \exists P(Y|\boldsymbol{x}) \in \mathscr{P}(Y|\boldsymbol{x}) \text{ s.t. } y = y^P_\ell\}.$$

- Computation: Solving linear programs [11], Iterating over the extreme points of $\Theta$ [11], exploiting the properties of NCC [3].

# NCC: Comments

NCC inherits properties of IDM [3]:

- May lead to **set-valued predictions**
- $\epsilon$-regularization can avoid **not well-defined** $P(y|\boldsymbol{x})$
- May provide reliable interval probabilities when seeing small numbers of observations
  - $n_k$: Number of training instances with label $y^k$
  - $n_k^{m,q_m}$: Number of training instances with label $y^k$ and feature $X^m$ takes value $x^{m,q_m}$

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- Provide tools to (naturally) take into account missing/partial data
  - Naive solutions are computationally expensive (in exponential time)
  - More efficient (polynomial-time) procedure exists

# NCC: Technical Details + Performance

Journal of Machine Learning Research 9 (2008) 581-621 Submitted 1/07; Revised 2/08; Published 4/08

## Learning Reliable Classifiers From Small or Incomplete Data Sets: The Naive Credal Classifier 2

**Giorgio Corani** GIORGIO@IDSIA.CH
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**Editor:** Charles Elkan

# Outline


- Graphical Interpretation of Probabilistic Models
- Naïve Bayesian/Credal classifiers
- Decision Trees
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# Discriminative Models

**Probabilistic Models**:

- Estimate $P(Y, \boldsymbol{X})$
- Chain rule (probability):

$$P(Y, \boldsymbol{X}) = P(Y|\text{pa}(Y)) \prod_{m=1}^{M} P(X^m|\text{pa}(X^m)).$$

**Extreme Cases**:

- Discriminative models: $Y \notin \text{pa}(X^m)$, $m \in [M]$
- Generative models: $\text{pa}(Y) = \emptyset$ and $Y \in \text{pa}(X^p)$, $m \in [M]$.

**Model Families**:

- How to encode/parametrize $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$.
- How to estimate $P(Y, \boldsymbol{X}) = P(Y|\boldsymbol{X})P(\boldsymbol{X})$ from training data.

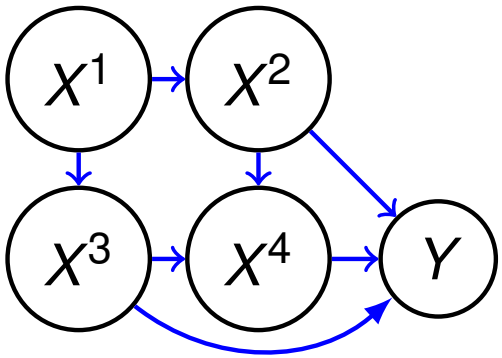
# Discriminative Models: Structure

Let's start with an example where one wishes to model

$$P(Y, \boldsymbol{X}) = P(Y, X^1, X^2, X^3, X^4).$$

Chain rule (probability):

$$P(Y, \boldsymbol{X}) = P(Y|\text{pa}(Y)) \prod_{m=1}^{M} P(X^m|\text{pa}(X^m)).$$




- $\text{pa}(Y) = \{X^2, X^3, X^4\}$
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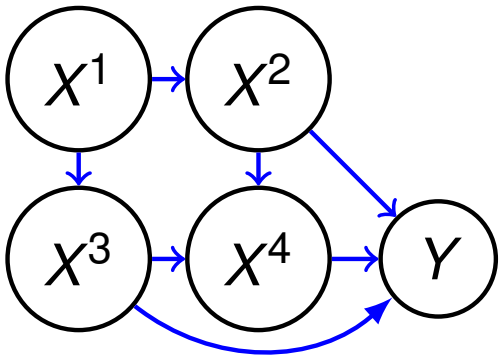


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The chain rule gives us

$$P(Y, \boldsymbol{X}) = P(Y|X^2, X^3, X^4)P(X^1)P(X^2|X^1)P(X^3|X^1)P(X^4|X^2, X^3).$$

# Classification Task

Prove that

$$P(y|\boldsymbol{x}) = P(y|\text{pa}(y))\,. \tag{18}$$

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$$P(y|\boldsymbol{x}) = P(y|\text{pa}(y)). \tag{18}$$

We have

$$P(y|\boldsymbol{x}) = \frac{P(y,\boldsymbol{x})}{\sum_{y' \in \mathcal{Y}} P(y',\boldsymbol{x})} \tag{19}$$

$$= \frac{P(y|\text{pa}(y)) \prod_{m=1}^{M} P(x^m|\text{pa}(x^m))}{\sum_{y' \in \mathcal{Y}} P(y'|\text{pa}(y')) \prod_{m=1}^{M} P(x^m|\text{pa}(x^m))} \tag{20}$$

$$= \frac{\prod_{m=1}^{M} P(x^m|\text{pa}(x^m)) P(y|\text{pa}(y))}{\prod_{m=1}^{M} P(x^m|\text{pa}(x^m)) \sum_{y' \in \mathcal{Y}} P(y'|\text{pa}(y'))} \tag{21}$$

$$= \frac{P(y|\text{pa}(y))}{\sum_{y' \in \mathcal{Y}} P(y'|\text{pa}(y'))} \tag{22}$$

$$= P(y|\text{pa}(y)). \tag{23}$$

# Classification Task: Comments

$P(Y|\boldsymbol{X})$ is extracted using **Bayes' theorem**

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- To solve the classification task, we only need $P(Y|\text{pa}(Y))$
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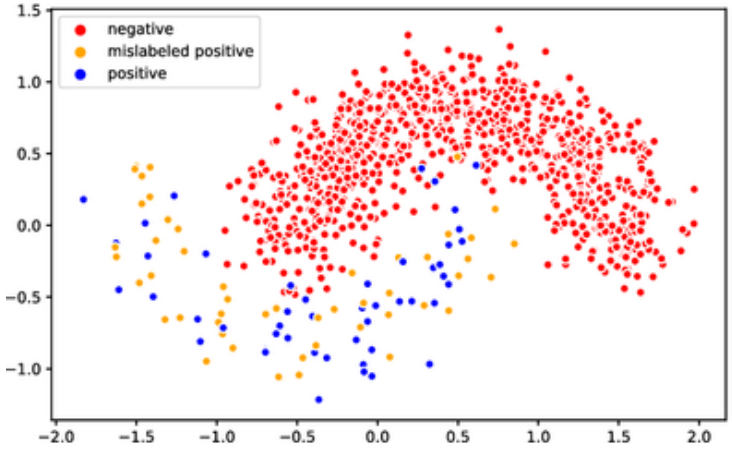
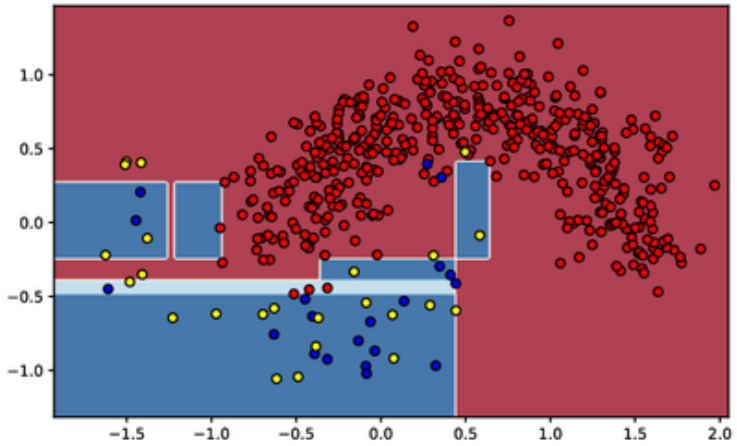
- To solve the classification task, we only need $P(Y|\text{pa}(Y))$
- Commonly used assumption: $\text{pa}(Y) = (X^1, \ldots, X^M)$
- $P(Y|\text{pa}(Y))$ can be defined either globally or locally:
  - Logistic regression, neural nets, etc., define $P(Y|\text{pa}(Y))$ globally
  - Decision tree, model trees, etc., define $P(Y|\text{pa}(Y))$ locally
  - Decision tree does not require $\text{pa}(Y) = (X^1, \ldots, X^M)$

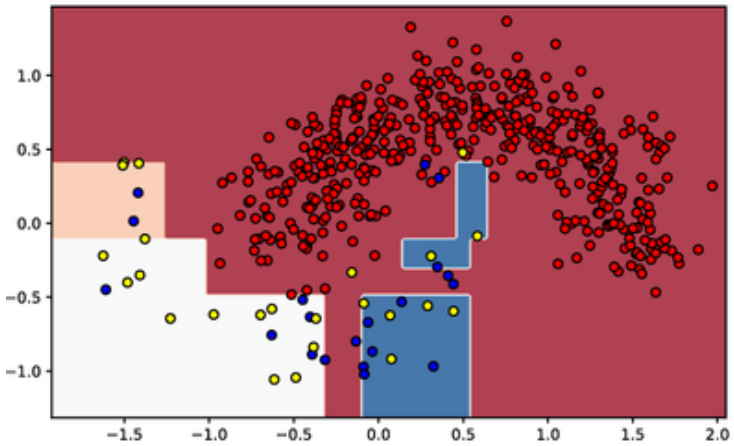
# Decision Trees: Example [12]



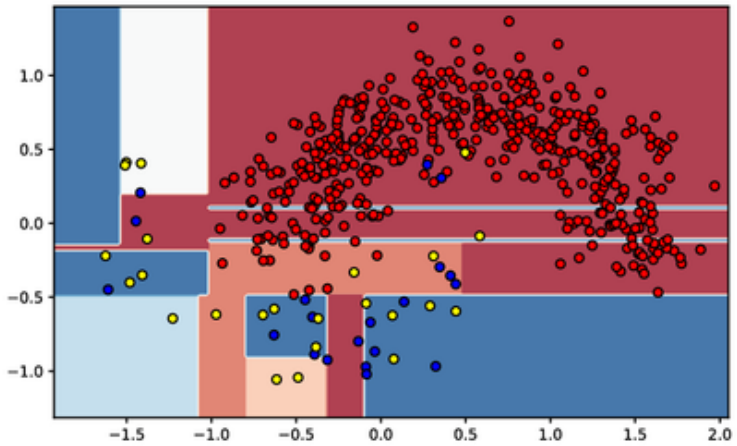

(a) Half-moons data set (ground truth)



(b) PU-Hellinger Decision Tree on the test set



(c) Hellinger Decision Tree on the test set



(d) CART on the test set

# Decision Trees: (Informal+Probabilistic) Definition

A decision trees is

- a collection of non-overlapping leaves $L_1, \ldots, L_H$
- where $L_1 \cap \ldots \cap L_H = \mathscr{X}$
- and each leaf $L_h$ has its own $P_h(Y|\text{pa}(Y))$

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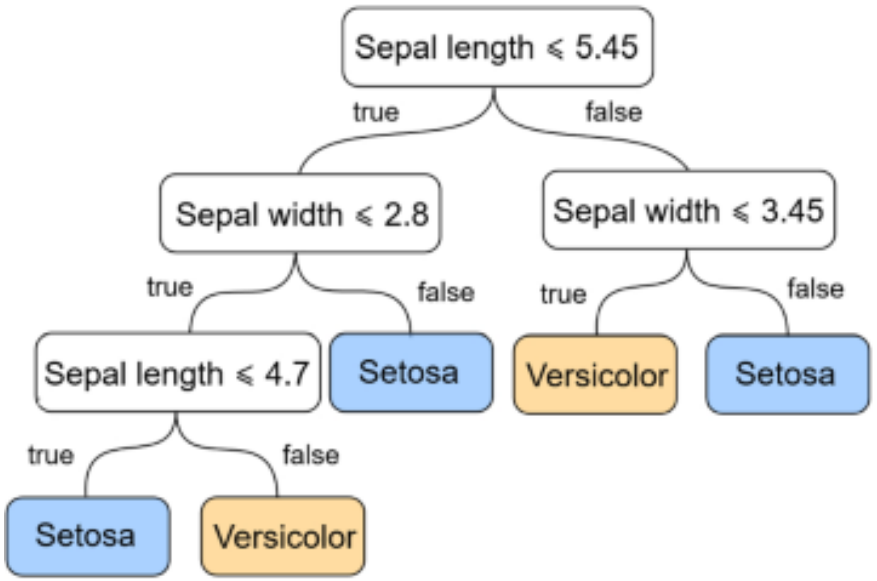
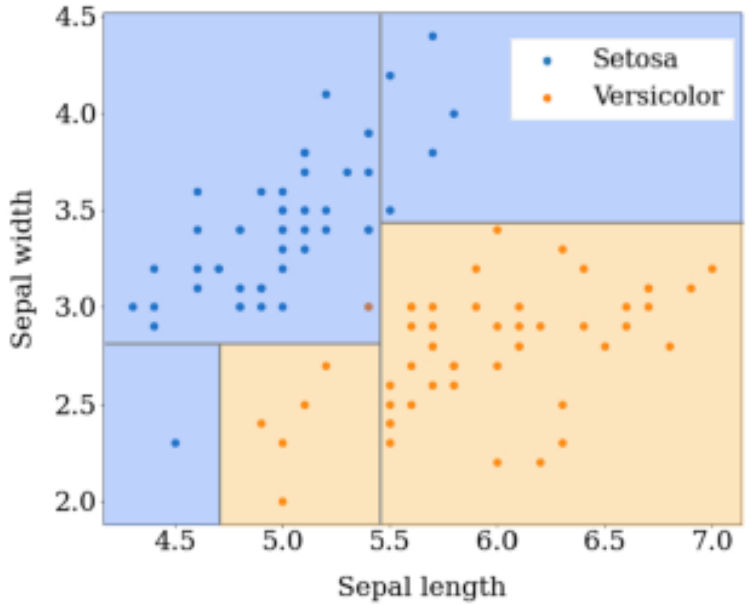
Learning an optimal decision tree from training data

- can be extremely hard (due to huge numbers of possible trees)
- and is often done approximately (top-down induction, bottom-up induction, etc.)

# Top-Down Induction (Example) [4]




**(a)** Tree visualization




**(b)** Partitioning visualization

# Top-down induction: Steps

**Basic Setup**:

- Training data $\boldsymbol{D} = \{(y^n, \boldsymbol{x}^n) | n \in [N]\}$
- Local hypothesis space $P(Y|\text{pa}(Y)) \in \mathscr{P}(Y|\text{pa}(Y))$
- An uncertainty measure $U$ or a loss function $\ell \longleftarrow$ assess how good/bad each **local classifier** is

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**Induction protocol**:

- Recursively partition the feature space $\mathscr{X}$
- From the current node, choose the best split which improves the evaluation criterion
- Evaluation criteria: Information gain, entropy, Gini score, etc.,
- Stopping criteria: No more gain on evaluation criterion $U$ or $\ell$

# Splitting Criteria: Entropy

- Entropy of each intermediate node $\boldsymbol{D}_n \subset \boldsymbol{D}$ with $P(Y|\boldsymbol{D}_n)$

$$U_E(P(Y|\boldsymbol{D}_n)) = -\sum_{y \in \mathcal{Y}} P(y|\boldsymbol{D}_n) \log_2 (P(y|\boldsymbol{D}_n)) .$$

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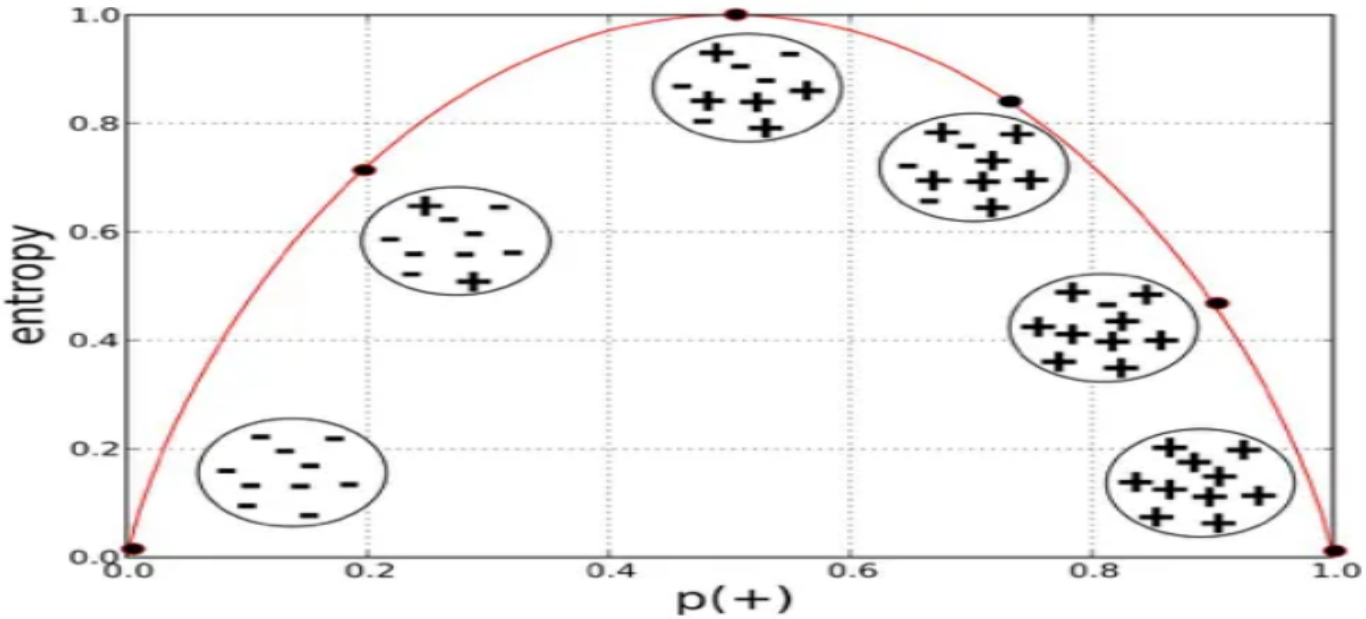

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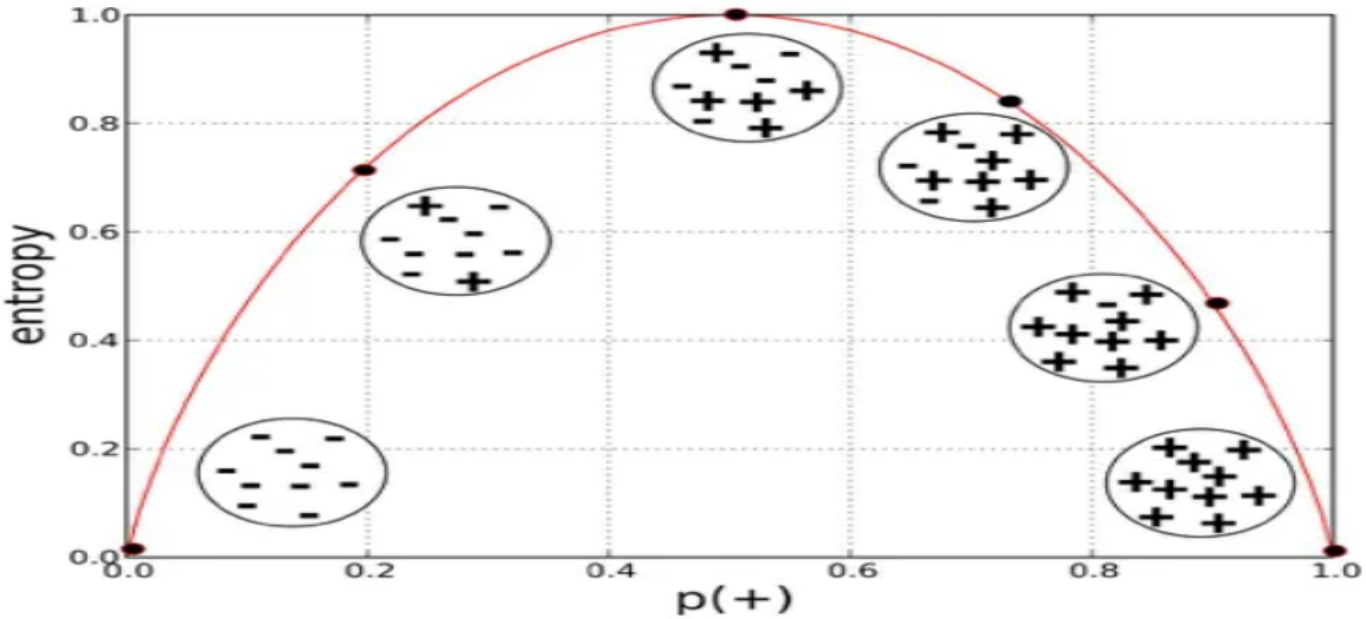


- For each possible split $\boldsymbol{D}_n = \boldsymbol{D}_n^1 \cup \boldsymbol{D}_n^2$, its entropy is

$$U_E(\boldsymbol{D}_n^1 \cup \boldsymbol{D}_n^2) = U_E(P(Y|\boldsymbol{D}_n)) + U_E(P(Y|\boldsymbol{D}_n)) .$$

# Compute Entropy

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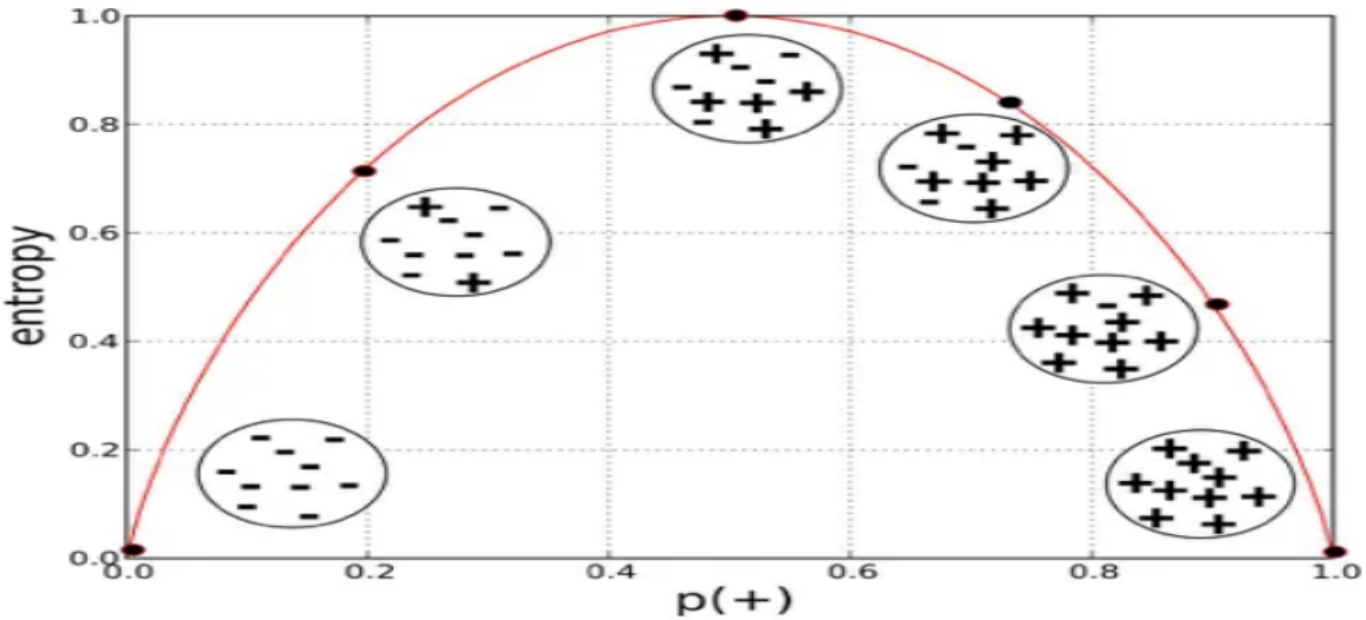


- Entropy of the bottom left and right nodes is 0
- Entropy of the top node is $-0.5\log_2(0.5) + 0.5\log_2(0.5) = 1$

# Splitting Criteria: Bayesian

In principle, we can employ Dirichlet models (DM) to

- derive Bayesian estimates of $P(Y|\boldsymbol{D}_n)$ and/or $U(P(y|\boldsymbol{D}_n))$
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- I haven't seen such decision trees
- I hope I can find some reference soon ...

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# Where and How to be Imprecise?

In principle, we can employ Imprecise Dirichlet models (IDM) to

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Credal Decision Trees [1, 8]

- Use IDM to derive interval estimates $\mathscr{P}(Y|\boldsymbol{D}_n)$ of $P(Y|\boldsymbol{D}_n)$
- Seek the highest entropy

$$U(\mathscr{P}(Y|\boldsymbol{D}_n)) = \max_{P \in \mathscr{P}} U(P(Y|\boldsymbol{D}_n)). \tag{25}$$

- **Note**: Each leaf is equipped a $P(Y|\boldsymbol{D}_h)$ → Precise predictions.

# Credal Decision Trees: Performance [8]

| Splitting criterion | 0% noise | 10% noise | 20% noise |
|---|---|---|---|
| Info-Gain (IG) | 78.96 | 77.49 | 74.76 |
| Info-Gain Ratio (IGR) | 78.97 | 77.66 | 75.14 |
| Imprecise Info-Gain (IIG) | 79.56 | 78.65 | 76.72 |
| Complete IIG (CIIG) | **79.63** | **78.66** | **76.74** |

Table: $10 \times 10$-fold cross-validation procedure: Average accuracies (on 60 data sets) with random noise to the features and the class variable

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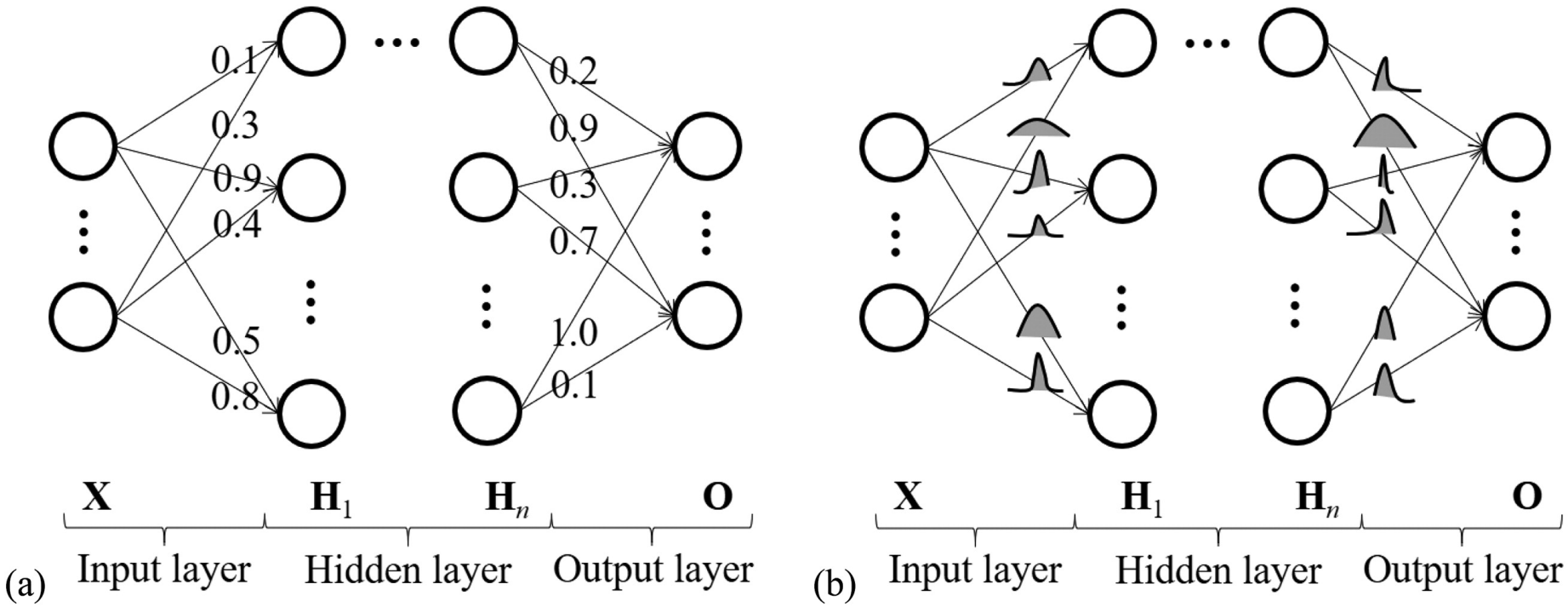
# Artificial neural networks vs Bayesian Neural Networks




Graphical interpretation of (a) ANN and (b) BNN

# Inference Problems [7]

**Algorithm 1** **Inference procedure for a BNN.**

Define $p(\boldsymbol{\theta}|D) = \dfrac{p(D_{\boldsymbol{y}}|D_{\boldsymbol{x}},\boldsymbol{\theta})\,p(\boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} p(D_{\boldsymbol{y}}|D_{\boldsymbol{x}},\boldsymbol{\theta}')\,p(\boldsymbol{\theta}')d\boldsymbol{\theta}'}$;

**for** $i = 0$ **to** $N$ **do**

  Draw $\boldsymbol{\theta}_i \sim p(\boldsymbol{\theta}|D)$;

  $\boldsymbol{y}_i = \Phi_{\boldsymbol{\theta}_i}(\boldsymbol{x})$;

**end for**

**return** $Y = \{\boldsymbol{y}_i | i \in [0, N)\}$, $\Theta = \{\boldsymbol{\theta}_i | i \in [0, N)\}$;

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**return** $Y = \{\boldsymbol{y}_i | i \in [0, N)\}$, $\Theta = \{\boldsymbol{\theta}_i | i \in [0, N)\}$;

- We need some way to aggregate the set of outputs

# Aggregation procedures

**Predict-then-aggregate** (You can try it yourself):

- For each Monte Carlo sample, turn $\Theta_\theta(\mathbf{x})$ into a hard prediction $y$.
- Aggregate the set of hard predictions into the final hard prediction.
- You might want to try with MLE (Frequentist), DM (Bayesian), IDM (IP), etc.

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**Aggregation-then-predict** (You can try it yourself):

- For each Monte Carlo sample, compute a soft prediction $\Theta_\theta(\mathbf{x})$.
- Aggregate the set of soft predictions into either
  - the final hard prediction
  - or a credal set, from which IP decision rules can be applied.

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Original software publication

# *BNNpriors*: A library for Bayesian neural network inference with different prior distributions

Vincent Fortuin [a,1,*], Adrià Garriga-Alonso [b,1], Mark van der Wilk [c,2], Laurence Aitchison [d,2]

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[c] Imperial College London, London, UK
[d] University of Bristol, Bristol, UK

ARTICLE INFO

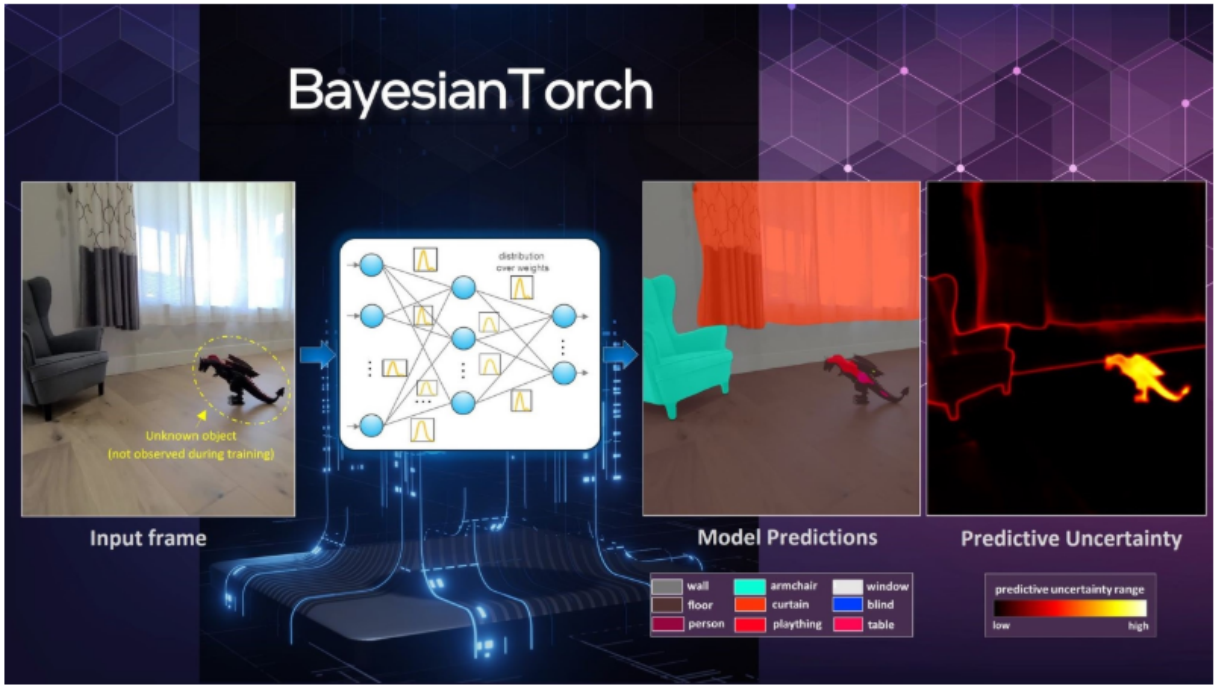
*Keywords:*
Machine learning
Bayesian neural networks
Prior distributions

ABSTRACT

Bayesian neural networks have shown great promise in many applications where calibrated uncertainty estimates are crucial and can often also lead to a higher predictive performance. However, it remains challenging to choose a good prior distribution over their weights. While isotropic Gaussian priors are often chosen in practice due to their simplicity, they do not reflect our true prior beliefs well and can lead to suboptimal performance. Our new library, *BNNpriors*, enables state-of-the-art Markov Chain Monte Carlo inference on Bayesian neural networks with a wide range of predefined priors, including heavy-tailed ones, hierarchical ones, and mixture priors. Moreover, it follows a modular approach that eases the design and implementation of new custom priors. It has facilitated foundational discoveries on the nature of the cold posterior effect in Bayesian neural networks and will hopefully catalyze future research as well as practical applications in this area.

# Improve Trustworthiness in Deep Learning Models with Bayesian-Torch

What is Bayesian Deep Learning?
Uncertainty Estimation in Deep Learning
Creating the Foundation for Robust, Trustworthy AI
A Framework for Seamless Bayesian Model Development
How to Use Bayesian-Torch
Model Inferencing and Uncertainty Estimation
Use Case: Medical Application (Colorectal Histology Diagnosis)
Accounting for Distributional Shifts
Advancing Real-World Benchmarks
Developing Efficient Computing Systems for BDL Models
Get Involved
About the Author



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# Probabilistic Models [10]

**Probabilistic Graphical Models**:

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**Extreme Cases**:

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**Model Families**:

- How to encode/parametrize $P(Y|\text{pa}(Y))$ and $P(X^m|\text{pa}(X^m))$.
- How to estimate $P(Y, \boldsymbol{X})$ from training data.

# Credal (Imprecise) Graphical Models

## Basic setup

- A set of features $\boldsymbol{X} = \{X^1, \ldots, X^M\}$
- A class variable $Y$ whose outcome $y \in \mathscr{Y}$

## Credal Models:

- $\mathscr{P} := \{P(Y, \boldsymbol{X}) | P \text{ is compatible with knowledge/data}\}$
- Chain rule (probability):

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# Beyond Multi-Class Classification

**Other predictive tasks**:

- Multi-Label Classification
- Multi-Dimensional Classification
- Multi-Target Prediction

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**Other predictive tasks**:

- Multi-Label Classification
- Multi-Dimensional Classification
- Multi-Target Prediction

**Examples**:

- Predict multiple diseases (yes, no) given ChetXray and Demographic information.
- Predict antimicrobial resistance (AMR) phenotypes (susceptible, intermediate, resistant) of multiple drugs given genomic sequences of the strain.
- Predict multiple characteristics of the object that appears in each grid cell: object (no, pedestrian, car, bicycle, and so on), moving (no, forward, backward, left, right), attention (yes, no), and so on.

# Probabilistic Models [10]

**Probabilistic Graphical Models**:

- Estimate $P(\boldsymbol{Y}, \boldsymbol{X})$, where $\boldsymbol{Y} = \{Y^1, \ldots, Y^K\}$
- Chain rule (probability):

$$P(\boldsymbol{Y}, \boldsymbol{X}) = \prod_{k=1}^{K} P(Y^k | \text{pa}(Y^k)) \prod_{m=1}^{M} P(X^m | \text{pa}(X^m)).$$

**Extreme Cases**:

- Discriminative models: $Y^k \notin \text{pa}(X^m)$, $k \in [K]$ and $m \in [M]$.
- Generative models: $\text{pa}(Y^k) \cap \boldsymbol{X} = \emptyset$, $k \in [K]$ and $m \in [M]$.

# Probabilistic Models [10]

**Probabilistic Graphical Models**:

- Estimate $P(\boldsymbol{Y}, \boldsymbol{X})$, where $\boldsymbol{Y} = \{Y^1, \ldots, Y^K\}$
- Chain rule (probability):

$$P(\boldsymbol{Y}, \boldsymbol{X}) = \prod_{k=1}^{K} P(Y^k | \text{pa}(Y^k)) \prod_{m=1}^{M} P(X^m | \text{pa}(X^m)).$$

**Extreme Cases**:

- Discriminative models: $Y^k \notin \text{pa}(X^m)$, $k \in [K]$ and $m \in [M]$.
- Generative models: $\text{pa}(Y^k) \cap \boldsymbol{X} = \emptyset$, $k \in [K]$ and $m \in [M]$.

**Model Families**:

- How to encode/parametrize $P(Y^k | \text{pa}(Y^k))$ and $P(X^m | \text{pa}(X^m))$.
- How to estimate $P(\boldsymbol{Y}, \boldsymbol{X})$ from training data.

# Credal (Imprecise) Graphical Models [5]

**Basic setup**

- A set of features $\boldsymbol{X} = \{X^1, \ldots, X^M\}$
- A set of class variables $Y^k$, whose outcome $y \in \mathscr{Y}$, $k = 1, \ldots, K$

**Credal Models**:

- $\mathscr{P} := \{P(\boldsymbol{Y}, \boldsymbol{X}) | P \text{ is compatible with knowledge/data}\}$
- Chain rule (probability):

$$P(\boldsymbol{Y}, \boldsymbol{X}) = \prod_{k=1}^{K} P(Y^k | \text{pa}(Y^k)) \prod_{m=1}^{M} P(X^m | \text{pa}(X^m)).$$

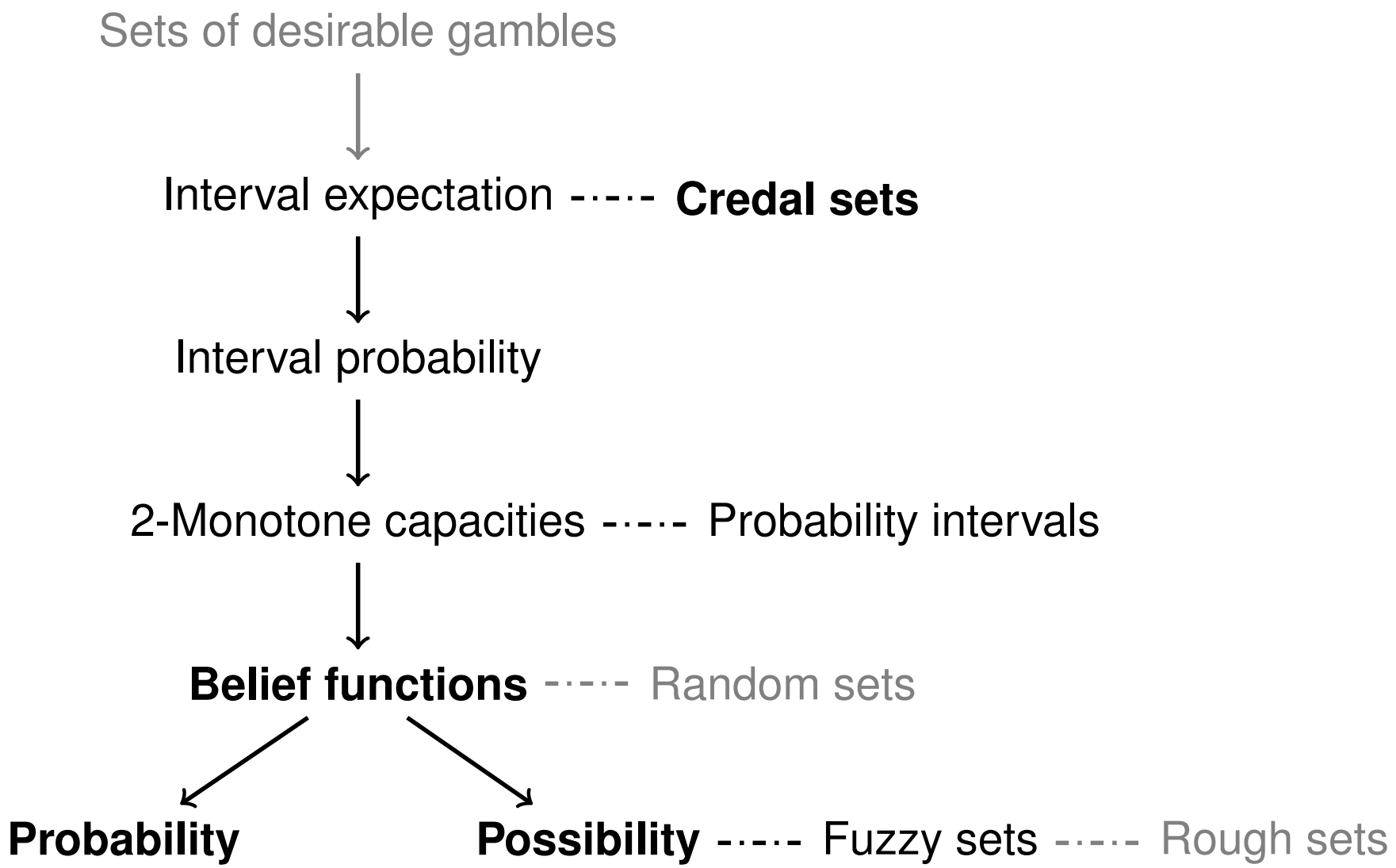
**Extreme Cases**:

- Discriminative models: $Y^k \notin \text{pa}(X^m)$, $k \in [K]$ and $m \in [M]$.
- Generative models: $\text{pa}(Y^k) \cap \boldsymbol{X} = \emptyset$, $k \in [K]$ and $m \in [M]$.

**Model Families**:

- How to encode/parametrize $P(Y^k | \text{pa}(Y^k))$ and $P(X^m | \text{pa}(X^m))$.
- How to estimate $\mathscr{P}(\boldsymbol{Y}, \boldsymbol{X})$ from training data.

# Other Families of Graphical Models



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