MT09 - TD #11 - Corrigé 

Ex 1 - $\dfrac{d\rho}{\rho(\rho_M-\rho)} = \dfrac{\sigma}{\rho_M}\,dt$ → Nécessite une décomposition en éléments simples de la fraction rationnelle.

1 - $\dfrac{1}{\rho(\rho_M-\rho)} = \dfrac{a}{\rho} + \dfrac{b}{\rho_M-\rho} = \dfrac{a\rho_M + \rho(b-a)}{\rho(\rho_M-\rho)} \rightarrow a = b = \dfrac{1}{\rho_M}$

2 - $\displaystyle\int_{\rho_0}^{\rho(t)} \frac{d\rho}{\rho(\rho_M-\rho)} = \frac{\sigma}{\rho_M}\int_0^t ds$

$$\rightarrow \frac{1}{\cancel{\rho_M}}\left\{[\ln\rho]_{\rho_0}^{\rho(t)} - [\ln(\rho_M-\rho)]_{\rho_0}^{\rho(t)}\right\} = \frac{1}{\cancel{\rho_M}}\sigma(t-0)$$

$$\rightarrow \ln\left[\frac{\rho(t)}{\rho_0}\ \frac{\rho_M-\rho_0}{\rho_M-\rho(t)}\right] = \sigma t$$

$$\rho(t) = \frac{1}{\frac{\rho_M}{\rho_0}-1}\left[\rho_M-\rho(t)\right]e^{\sigma t}$$

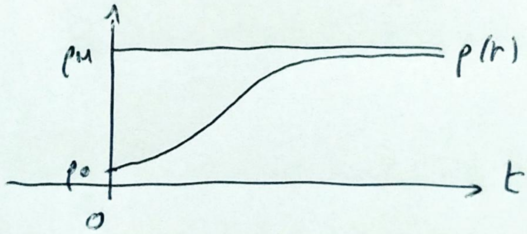
$$\rho(t)\cdot\left\{1+\frac{1}{\frac{\rho_M}{\rho_0}-1}e^{\sigma t}\right\} = \frac{\rho_M e^{\sigma t}}{\frac{\rho_M}{\rho_0}-1}$$

$$\rho(t)\cdot\left(\frac{\rho_M}{\rho_0}-1+e^{\sigma t}\right) = \rho_M e^{\sigma t}$$

$$\rightarrow \rho(t) = \rho_M\cdot\frac{e^{\sigma t}}{\frac{\rho_M}{\rho_0}-1+e^{\sigma t}}, \quad t \geq 0.$$

3 - $\rho_M > \rho_0$, donc $\dfrac{\rho_M}{\rho_0} - 1 > 0 \rightarrow \rho(t) \leq \rho_M$.

$\lim\limits_{t\to\infty} \rho(t) = \rho_M^-$.



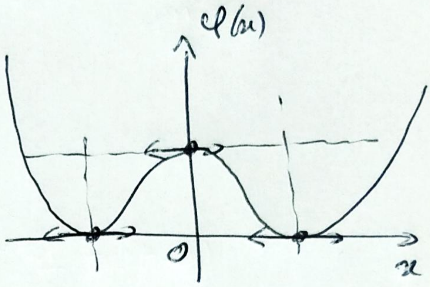

4 - $\rho(\tau) = \dfrac{\cancel{\rho_M}}{2} = \cancel{\rho_M}\dfrac{e^{\sigma\tau}}{\frac{\rho_M}{\rho_0}-1+e^{\sigma\tau}} \rightarrow \dfrac{1}{2}\left(\dfrac{\rho_M}{\rho_0}-1+e^{\sigma\tau}\right) = e^{\sigma\tau}$

$$\rightarrow \frac{1}{2}\left(\frac{\rho_M}{\rho_0} - 1\right) = \frac{1}{2} e^{\sigma\tau}$$

$$\rightarrow \tau = \frac{1}{\sigma} \ln\left(\frac{\rho_M}{\rho_0} - 1\right).$$

**Ex 2** – 1– $\varphi'(x) = 4x(x^2-1)$ , $\varphi''(x) = 12x^2 - 4$ .

| $x$ | $-\infty$ | | $-1$ | | $0$ | | $1$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $\varphi'$ | | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ | |
| $\varphi'$ | $+\infty$ | $\searrow$ | $0$ | $\nearrow$ | $1$ | $\searrow$ | $0$ | $\nearrow$ | $+\infty$ |




2 – $$\tilde{y} = \begin{pmatrix} x \\ v \end{pmatrix}, \quad \frac{d\tilde{y}(t)}{dt} = \begin{pmatrix} v(t) \\ -\varphi'(x(t)) - a\, v(t) \end{pmatrix} := \vec{f}(\tilde{y}(t)).$$

Le système est autonome.

3 – $\dot{x} = +v \longrightarrow$ $\varphi'(x(t))\, \dot{x}(t) = +\varphi'(x(t)) \cdot v(t)$

$$\frac{d}{dt}\varphi(x(t)) = \underline{+\varphi'(x(t)) \cdot v(t)}$$

$\dot{v} = -\varphi'(x) - a v \rightarrow$ $v(t)\dfrac{dv(t)}{dt} = \underline{-v(t).\varphi'(x(t))} - a\, v^2(t)$

$$\frac{d}{dt}\left(\frac{v^2(t)}{2}\right) = \quad '' \qquad\qquad ''.$$

En sommant,

$$\frac{dE(t)}{dt} = -a\, v^2(t) \leq 0.$$

4 – États d'équilibre : $\tilde{y} \,/\, \tilde{f}(\tilde{y}) = 0 \iff \begin{cases} v = 0 \\ -\varphi'(x) - av = 0 \end{cases}$

$$\iff \begin{cases} v = 0 \\ -4x(x^2-1) - av = 0 \end{cases} \implies$$

$(x, v) = (0, 0)$

ou

$(x, v) = (-1, 0)$

ou

$(x, v) = (1, 0)$.

5 – • Considérons d'abord $\tilde{y}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (Q7. traitée en même temps) 3°]

$$f(y) \simeq f(y^*) + Df(y^*)(y - y^*)$$

$$A := Df(y^*) = \begin{bmatrix} 0 & 1 \\ -\varphi''(0) & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -a \end{bmatrix}$$

$\mathrm{tr}\, A = -a$ $\lambda^2 + a\lambda - 4 = 0$

$\det A = -4$ $\Delta = a^2 + 16 \rightarrow \lambda_\pm = \dfrac{-a \pm \sqrt{a^2+16}}{2}$

Il existe une valeur propre strictement positive → l'équilibre est instable.

• Considérons maintenant $\tilde{y}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = Df(y^*) = \begin{bmatrix} 0 & 1 \\ -8 & -a \end{bmatrix}$$

$\mathrm{tr}\, A = -a$

$\det A = 8$

$\lambda^2 + a\lambda + 8 = 0$

$\Delta = a^2 - 32$

a) Si $a \geq \sqrt{32}$, $\Delta > 0$, $\lambda = \dfrac{-a \pm \sqrt{a^2-32}}{2}$ → $\lambda_1, \lambda_2 < 0$ stable.

b) Si $a < \sqrt{32}$, $\Delta < 0$, $\lambda = \dfrac{-a \pm i\sqrt{32-a^2}}{2}$, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$ stable

→ Equilibre stable.

• Idem pour $\tilde{y}^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. stable.

6 – $\dfrac{dE(t)}{dt} = 0$ → L'énergie est conservée (on dit que le système est conservatif).

$$\rightarrow E(t) = E(0) = x_0 + \frac{[v(0)]^2}{2}$$

Ou bien le système est au repos ($x = \pm 1$, $v = 0$), ou bien il est en mouvement perpétuel.

Remarque :

$$\begin{cases} \dfrac{d}{dr} \varphi(u(r)) = + \varphi'(u(r)) \cdot v(r) \\ \\ \dfrac{d}{dr} (v^2(r)/2) = - \varphi'(u(r)) \cdot v(r) \end{cases}$$

↑ Travail de la force de gravité ↓ (gradient du potentiel)

7- Voir 5-

Ex 3- 1- $\vec{y} = \begin{pmatrix} p \\ q \end{pmatrix}$ $\quad \dfrac{d\vec{y}}{dr} = \begin{pmatrix} [\alpha - q(r)\cdot\beta]\cdot p(r) \\ [\delta\, p(r) - \gamma]\cdot q(r) \end{pmatrix} := \vec{f}(\vec{y})$.

Le système est autonome.

On doit résoudre $\vec{f}(\vec{y}) = \vec{0}$ : $\begin{cases} (\alpha - \beta q)\cdot p = 0 \\ (\delta p - \gamma)\cdot q = 0 \end{cases}$

$\rightarrow \begin{cases} p^* = 0 \\ q^* = 0 \end{cases}$ ou $\begin{cases} p^* = \gamma/\delta \\ q^* = \alpha/\beta \end{cases}$.

a) Traitons d'abord : $\vec{y}^* = (0,0)$

$\vec{f}(y) \simeq \vec{f}(0) + \begin{bmatrix} \alpha - \beta q & -\beta p \\ \delta q & \delta p - \gamma \end{bmatrix}\Bigg|_{(p,q)=(0,0)}$ ; $\quad A = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$

$\Rightarrow$ 1 valeur propre <u>positive</u> $\Rightarrow$ instable.

b) $\vec{y}^* = \left(\dfrac{\gamma}{\delta}, \dfrac{\alpha}{\beta}\right) \rightarrow A = \begin{bmatrix} 0 & -\beta\gamma/\delta \\ \dfrac{\delta\alpha}{\beta} & 0 \end{bmatrix}$

$\begin{cases} \operatorname{tr} A = 0 \\ \det A > 0 \end{cases}$ $\rightarrow$ Valeurs propres imaginaires pures.

$\rightarrow$ <u>centre</u>.

3- $\begin{cases} \dot p = (1-q)p \\ \dot q = (p-1)q \end{cases} \rightsquigarrow \dot{\overline{p+q}} = p-q$

Par ailleurs, $\dot{\overline{\ln p}} = \dfrac{\dot p}{p} = 1-q \rightarrow \dot{\overline{(\ln p + \ln q)}} = p-q$

$\dot{\overline{\ln q}} = \dfrac{\dot q}{q} = p-1$

$\rightarrow \dfrac{d}{dt}\left(p - \ln p + q - \ln q\right) = 0$

$\rightarrow \eta(p(t), q(t)) = \eta(p(0), q(0)) \quad \forall t \geq 0.$

La grandeur $\eta$ est conservée.

4- $A^T = -A$ : $(A^T)_{ij} = (-A)_{ij} \Longleftrightarrow a_{ij} = -a_{ji}$

En particulier : $a_{ii} = -a_{ii} \rightsquigarrow 2a_{ii} = 0 \rightsquigarrow a_{ii} = 0 \; \forall i.$

Soit $q(x) = x^T A x$. $q(x) = \langle x, Ax \rangle = \langle A^T x, x \rangle$

$= \langle -Ax, x \rangle$

$= -\langle Ax, x \rangle$

$= -\langle x, Ax \rangle$

$= -q(x)$

$\rightsquigarrow 2q(x) = 0 \rightsquigarrow q(x) = 0.$

$\underline{x^T A x = 0.}$

5- $\vec\nabla \eta(\vec y) = \begin{pmatrix} 1 - 1/p \\ 1 - 1/q \end{pmatrix}$. $\Rightarrow pq \cdot \vec\nabla\eta(y) = \begin{pmatrix} q(p-1) \\ (q-1)p \end{pmatrix}$

$\Rightarrow A\,(pq\,\vec\nabla\eta(y)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q(p-1) \\ (q-1)p \end{pmatrix} = \begin{pmatrix} (1-q)p \\ (p-1)q \end{pmatrix} = \vec F(\vec y)$

On a $\dfrac{d\vec y}{dt} = pq \; A \; \vec\nabla\eta(\vec y)$

mais $\dfrac{d}{dt}(\eta(y)) = \sum_{i=1}^{2} \dfrac{\partial \eta}{\partial y_i} \dfrac{dy_i}{dt} = \dfrac{\partial\eta}{\partial p}\dfrac{dp}{dt} + \dfrac{\partial \eta}{\partial q}\dfrac{dq}{dt} = \vec\nabla\eta(\vec y) \cdot \dfrac{d\vec y}{dt}$

$\rightsquigarrow \dot\eta(y) = pq \; \nabla\eta(y)^T A \nabla\eta(y) = 0$ d'après 4. $\blacksquare$