

Ex 1

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \\ (0) & & a_{nn} \end{bmatrix} = D - F, \quad F = \begin{bmatrix} & & -a_{1n} \\ & & \\ (0) & & \end{bmatrix}$$

1- $a_{ii} \bar{x}_i = b_i, \quad 2 \leq i \leq n \rightarrow \bar{x}_i = b_i / a_{ii}, \quad 2 \leq i \leq n.$

et $a_{11} \bar{x}_1 + a_{1n} \bar{x}_n = b_1 \rightarrow \bar{x}_1 = (b_1 - a_{1n} \frac{b_n}{a_{nn}}) / a_{11}.$

2- Jacobi: $\begin{cases} D x^{(k+1)} = F x^{(k)} + b \\ x^{(0)} \text{ donné.} \end{cases}$

3- ~~$a_{ii} x_i^{(0)} = 0 + b_i, \quad 2 \leq i \leq n$~~
 $a_{ii} x_i^{(1)} = 0 + b_i, \quad 2 \leq i \leq n$
 $\Rightarrow \underline{x_i^{(1)} = b_i / a_{ii}, \quad 2 \leq i \leq n}$

et $a_{11} x_1^{(1)} = -a_{1n} x_n^{(0)} + b_1 \rightarrow \underline{x_1^{(1)} = (b_1 - a_{1n} x_n^{(0)}) / a_{11}}$

$$\rightarrow x^{(1)} - \bar{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{a_{1n}}{a_{11}} (x_n^{(0)} - b_n) \\ \vdots \end{bmatrix}.$$

4- $\underline{x_i^{(k)} = b_i / a_{ii}, \quad 2 \leq i \leq n}$

$a_{11} x_1^{(k)} = -a_{1n} x_n^{(k)} + b_1 \Rightarrow x_1^{(k)} = (b_1 - a_{1n} \frac{b_n}{a_{nn}}) / a_{11}.$
 $\quad \quad \quad \parallel \frac{b_n}{a_{nn}} \quad \quad \quad = \bar{x}_1.$

5- $D^{-1}F = \begin{bmatrix} 0 & & -a_{1n}/a_{11} \\ & \ddots & \\ (0) & & (0) \end{bmatrix} \Rightarrow \|D^{-1}F\|_{\infty} = \frac{|a_{1n}|}{|a_{11}|}.$
 $\rho(D^{-1}F) = 0.$

$$6 - D x^{(k+1)} = F x^{(k)} + b$$

$$D \bar{x} = F \bar{x} + b$$

$$D (x^{(k+1)} - \bar{x}) = F (x^{(k)} - \bar{x}) \rightarrow x^{(k+1)} - \bar{x} = D^{-1} F (x^{(k)} - \bar{x})$$

$$\text{En particulier, } x^{(k+1)} - \bar{x} = (D^{-1} F)^2 (x^{(k)} - \bar{x})$$

$$\text{mais } (D^{-1} F)^2 = 0 \text{ (nilpotente)} \Rightarrow \underline{x^{(k)} = \bar{x}}.$$

$$\boxed{\text{Ex 2}} \quad 1 - (D - E - F) x^{\text{err}} = b$$

$$\text{Jacobi: } D x^{(k+1)} = (E + F) x^{(k)} + b$$

$$x^{(k+1)} = D^{-1} (E + F) x^{(k)} + D^{-1} b.$$

$$2 - \text{Soit } C = D^{-1} (E + F), \quad \underline{C}_i = \frac{[D^{-1} (E + F)]_i}{1} = (D^{-1})_i (E + F)$$

$$\rightarrow \underline{C}_i = \left[0 \quad \dots \quad 0 \quad \frac{1}{a_{ii}} \quad 0 \quad \dots \quad 0 \right] \begin{bmatrix} \frac{E+F}{1} \\ \frac{E+F}{i} \\ \frac{E+F}{n} \end{bmatrix} = \frac{1}{a_{ii}} (E+F)_i.$$

$$A \text{ est DSD : } |a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \forall i.$$

$$\underline{C}_i = \frac{1}{a_{ii}} \left[a_{i1} \dots a_{i,i-1} \quad 0 \quad a_{i,i+1} \dots a_{in} \right]$$

$$\sum_i |C_{ij}| = \frac{\sum_{j \neq i} |a_{ij}|}{|a_{ii}|}$$

$$\Rightarrow \max_i \sum_j |C_{ij}| = \max_i \frac{\sum_{j \neq i} |a_{ij}| \text{ (DSD)}}{|a_{ii}|} < 1$$

$$\Rightarrow \|C\|_\infty < 1.$$

3 - cf Thm du cours.

Ex 3

$$x^{(k+1)} = C x^{(k)} + d, \quad C = P \wedge P^{-1}$$

3°)

$$x^{(k+1)} = P \wedge P^{-1} x^{(k)} + d$$

$$\Leftrightarrow P^{-1} x^{(k+1)} = \wedge P^{-1} x^{(k)} + d$$

$$P^{-1} \bar{x} = \wedge P^{-1} \bar{x} + d$$

$$\underbrace{P^{-1} [x^{(k+1)} - \bar{x}]}_{= e^{(k+1)}} = \wedge \underbrace{P^{-1} [x^{(k)} - \bar{x}]}_{= e^{(k)}} \Rightarrow \boxed{e^{(k+1)} = \wedge e^{(k)}}$$

$$\text{Soit } i_0 \in \{1, \dots, n\} / |d_{i_0}| = \rho(C) = 1$$

$$e^{(k+1)} = \wedge e^{(k)}$$

$$\begin{aligned} |e^{(k+1)}_{i_0}| &= |\wedge e^{(k)}_{i_0}| \\ &= \underbrace{|d_{i_0}|}_{=1} |e^{(k)}_{i_0}| = |e^{(k)}_{i_0}| \\ &= \dots = \|e^{(0)}_{i_0}\|. \end{aligned}$$

$\Rightarrow$ La méthode ne converge pas.

$$3 - e^{(k+1)} = \wedge e^{(k)}$$

$$\begin{aligned} |e^{(k)}_{i_0}| &= |e^{(k)}_{i_0}| \leq [\rho(C)]^k |e^{(0)}_{i_0}| \quad \forall i \\ &= (1-\epsilon)^k |e^{(0)}_{i_0}| \longrightarrow 0 \end{aligned}$$

mais convergence lente.

Ex 4

$$A = \begin{bmatrix} 2 & -1 & & (0) \\ -1 & & & -1 \\ & & & \\ (0) & & & -1 & 2 \end{bmatrix}$$

4°)

$$1 - x^T A x = 2x_1^2 - 2x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2 - \dots - 2x_{n-1}x_n + x_n^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \dots + (x_{n-1} - x_n)^2 + x_n^2 \geq 0$$

$$x^T A x = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x_i = x_{i+1} \quad \forall i \in \{1, \dots, n-1\} \\ x_n = 0 \end{cases} \Rightarrow x_i = 0 \quad \forall i$$

→ A sdp
→ A inversible

$$2 - A = L + L^T \text{ avec } L = \begin{pmatrix} 1 & & (0) \\ -1 & & \\ (0) & -1 & 1 \end{pmatrix}$$

$$L x^{(k+1)} = -L^T x^{(k)} + b$$

Écrire la méthode itérative en expressions algébriques.

$$4 - \text{Calcul de } L^{-1} : Lx = b \Leftrightarrow \begin{cases} x_1 = b_1 \\ -x_1 + x_2 = b_2 \\ -x_2 + x_3 = b_3 \\ \vdots \\ -x_{n-1} + x_n = b_n \end{cases}$$

$$\begin{cases} b_1 = x_1 \\ x_2 = x_1 + b_2 = b_1 + b_2 \\ x_3 = x_2 + b_3 = b_1 + b_2 + b_3 \\ \vdots \\ x_n = b_1 + b_2 + \dots + b_n \end{cases}$$

$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & & (0) \\ & 1 & \\ & & 1 \\ & & & 1 \end{bmatrix}$$

$$5 - C = -L^{-1}L^T = - \begin{bmatrix} 1 & & & (0) \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & (0) \\ 0 & 1 & & -1 \\ 0 & 0 & 1 & \\ (0) & & & -1 \end{bmatrix}$$

$$\Rightarrow C = - \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\|C\|_{\infty} = 2$$

on ne peut pas conclure.