

Ex 1

$$\begin{array}{c|ccc}
 x & -1 & 0 & 1 \\
 \hline
 g' & - & 0 & + \\
 \hline
 g & 0 & -1/3 & 0
 \end{array}$$

$$I = [-1, 1]$$

$$g(I) = [-1/3, 0] \subset I.$$

$$2 - |g'(x)| = \frac{2}{3} |x| \leq \frac{2}{3} < 1. \rightarrow \text{Contractant}$$

$$3 - \text{Thm: } \begin{cases} g \in \mathcal{C}^0, g(I) \subset I \\ g \text{ contractante sur } I \end{cases} \Rightarrow \exists ! x^* \in I / x^* = g(x^*)$$

$$\text{et la suite } \begin{cases} x_0 \in I \\ x_{n+1} = g(x_n) \end{cases} \text{ converge vers } x^*.$$

$$x^* = g(x^*) = \frac{x^{*2} - 1}{3} \rightarrow x^{*2} - 3x^* - 1 = 0. \Delta = 13. \rightarrow x^* = \frac{3 \pm \sqrt{13}}{2}$$

$$\sqrt{13} > \sqrt{9} = 3 \quad \frac{3+3}{2} = 3 > 1. \text{ Donc } x^* = \frac{3 - \sqrt{13}}{2} \in I.$$

$$\frac{3 + \sqrt{13}}{2} \notin I$$

$$4 - \text{TAF: } \exists \xi_n \in]-1, 1[/ g(x_n) - g(x^*) = g'(\xi_n) (x_n - x^*).$$

Plus explicitement :

$$\begin{aligned}
 \begin{cases} x_{n+1} = g(x_n) \\ x^* = g(x^*) \end{cases} &\rightarrow x_{n+1} - x^* = g(x_n) - g(x^*) \\
 &= \frac{x_n^2 - 1}{3} - \frac{x^{*2} - 1}{3} \\
 &= \frac{1}{3} (x_n^2 - x^{*2}) \\
 &= \frac{2}{3} \frac{x_n + x^*}{2} \cdot (x_n - x^*) \\
 &= \xi_n (x_n - x^*)
 \end{aligned}$$

$$\text{clairement } |\xi_n| \leq \frac{2}{3}.$$

$$\text{Par récurrence, } |x_n - x^*| \leq \left(\frac{2}{3}\right)^n |x_0 - x^*|.$$

$$\begin{aligned}
 5 - |x_{n+p} - x_n| &= |x_{n+p} - x_{n+p-1} + x_{n+p-1} - x_{n+p-2} + \dots \\
 &\quad + x_{n+2} - x_{n+1} + x_{n+1} - x_n| \\
 &\leq |x_{n+p} - x_{n+p-1}| + \dots + |x_{n+1} - x_n|
 \end{aligned}$$

2 -

Mais $\forall k, \quad |x_{k+1} - x_k| \leq \frac{2}{3} |x_k - x_{k-1}|$
 $\leq \dots \leq \left(\frac{2}{3}\right)^k |x_1 - x_0|$

$$\Rightarrow |x_{n+p} - x_n| \leq \left(\frac{2}{3}\right)^{n+p-1} |x_1 - x_0| + \dots + \left(\frac{2}{3}\right)^n |x_1 - x_0|$$

$$= \left(\frac{2}{3}\right)^n \cdot \left[\left(\frac{2}{3}\right)^{p-1} + \left(\frac{2}{3}\right)^{p-2} + \dots + \left(\frac{2}{3}\right) + 1 \right] |x_1 - x_0|$$

$$= \frac{1 - (2/3)^p}{1 - (2/3)} |x_1 - x_0|$$

$$\Rightarrow |x_{n+p} - x_n| \leq \left(\frac{2}{3}\right)^n \frac{1 - (2/3)^p}{1 - (2/3)} \cdot |x_1 - x_0|$$

En faisant tendre p vers l'infini,

$$|x_n - x_0| \leq \left(\frac{2}{3}\right)^n \cdot \frac{1}{1 - 2/3} |x_1 - x_0|$$

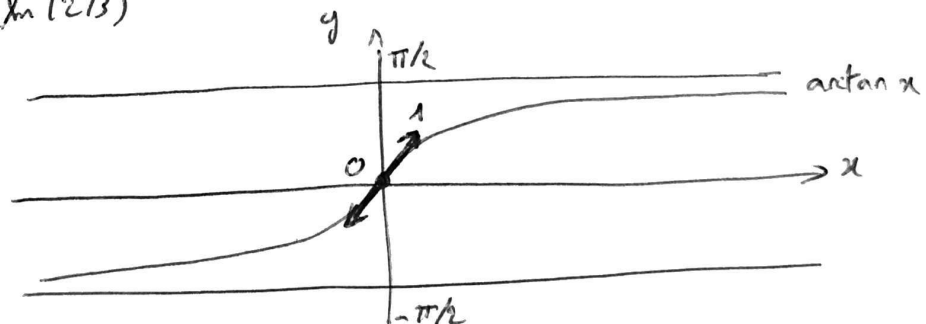
6 - Application : $x_0 = 0, x_1 = -1/3 \Rightarrow |x_1 - x_0| = \frac{1}{3}$

$$\Rightarrow |x_n - x_0| \leq \left(\frac{2}{3}\right)^n$$

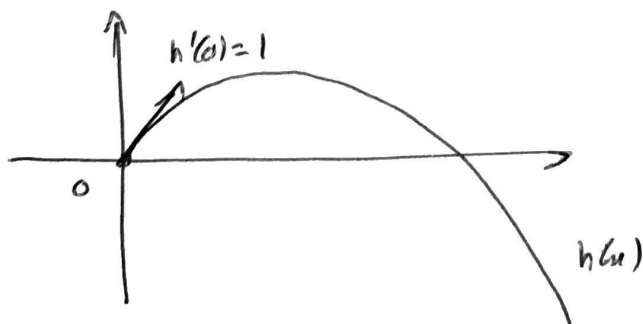
Condition suffisante : $\left(\frac{2}{3}\right)^n < 10^{-15} \xrightarrow{\log} n \ln\left(\frac{2}{3}\right) \leq \ln(10^{-15})$

$\Rightarrow n > \frac{\ln(10^{-15})}{\ln(2/3)} \approx 85,18$. On trouve $\boxed{n=86}$.

Ex 2 1.



2 -



$$h(x) \sim -\frac{\pi}{2} x^2 \text{ qd } x \rightarrow \infty$$

$$h'(0)=1 \Rightarrow \exists \delta > 0, \delta < 1, h(\delta) > 0$$

$$h(\delta) > 0, h(x) \xrightarrow{x \rightarrow +\infty} -\infty$$

$\Rightarrow$ Thm des valeurs intermédiaires :
 $\exists x^* / h(x^*) = 0$.

$h(-x) = -h(x) \rightarrow h$ est une fonction impaire.

Donc si x^* est racine ($h(x^*) = 0$), $(-x^*)$ est aussi racine.

3 - $f(x) = \operatorname{arctan} x$

$f'(x) = \frac{1}{1+x^2} \rightarrow$ Méthode de Newton : $\begin{cases} x_0 \in \mathbb{R}, \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \end{cases}$

soit $x_{n+1} = x_n - (1+x_n^2) \operatorname{arctan} x_n$.

En prenant $x_0 = x^*$, on a $x_1 = x^* - (1+x^{*2}) \operatorname{arctan} x^*$,

mais $h(x^*) = 0 = 2x^* - (1+x^{*2}) \operatorname{arctan} x^*$, donc

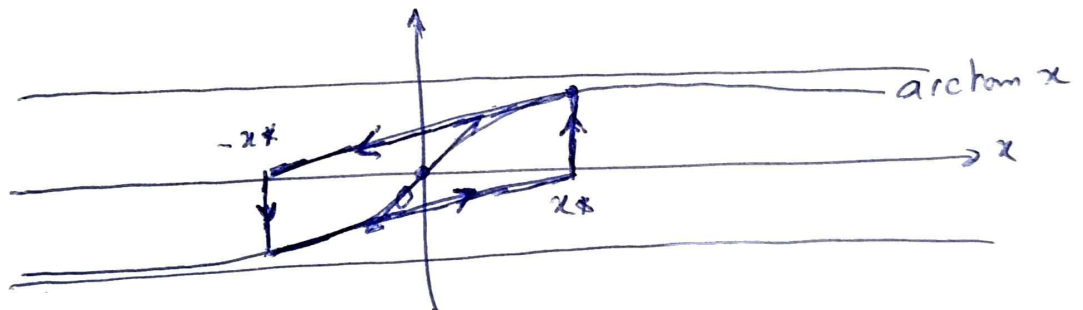
$$x^* - (1+x^{*2}) \operatorname{arctan} x^* = -x^*$$

$\rightarrow x_1 = -x^*$

$x_2 = x^*$

$x_3 = -x^*, \dots$

On tombe sur une suite alternée (non convergente)



4 - $x_{n+1} = x_n - (1+x_n^2) \operatorname{arctan} x_n$. On retrouve la méthode de Newton appliquée à la fonction $x \mapsto \operatorname{arctan} x$.

5 - $g'(x) = 1 - \operatorname{arctan} x \cdot (2x) - \frac{1+x^2}{1+x^2} = -2x \operatorname{arctan} x$.

on sait que $|\operatorname{arctan} x| \leq |x| \Rightarrow |g'(x)| \leq 2x^2$.

Pour $|x| \leq \beta$, on a $|g'(x)| \leq 2\beta^2 < 1$.

$\Rightarrow \begin{cases} g(\mathbb{R}) \subset \mathbb{R} \\ |g'(x)| \leq 2\beta^2 < 1 \end{cases} \rightarrow$ Contractant $\rightarrow \exists ! \hat{x} (=0) / g(\hat{x}) = \hat{x}$ et la méthode de Newton converge.

2 - Newton n'écrit donc :

$$x_{n+1} = g(x_n)$$

avec $g(x) = 2x - ax^2$.

On montre que pour $a \in]1, 2[$ et $x \in [\frac{1}{2}, 1]$, on a $|g'(x)| < 1$
 $\Rightarrow$ Contractant.

3 - $g(1) = 2 - a$.

$$\begin{aligned} 1 &< a < 2 \\ -2 &< -a < -1 \\ 0 &< 2-a < 1 \end{aligned}$$

Par exemple, pour $a = \frac{7}{4} (1 < 8)$, on a $g(1) = \frac{1}{4} \notin [\frac{1}{2}, 1]$.

On n'a pas toujours $g([\frac{1}{2}, 1]) \subset [\frac{1}{2}, 1]$ et on ne peut pas appliquer le théorème du cours. On ne peut pas conclure quant à la convergence.

4 - $x_* = \frac{1}{a}$.
$$\begin{aligned} x_{n+1} - x_* &= g(x_n) - g(x_*) \\ &= (2x_n - ax_n^2) - (2x_* - ax_*^2) \\ &= 2\left(x_n - \frac{1}{a}\right) - a\left[x_n^2 - \left(\frac{1}{a}\right)^2\right] \end{aligned}$$

$$\begin{aligned} \rightarrow a(x_{n+1} - x_*) &= \left(x_n - \frac{1}{a}\right) \cdot \left[2a - a^2 \cdot \left(x_n + \frac{1}{a}\right)\right] \\ &= \left(x_n - \frac{1}{a}\right) [a - a^2 x_n] = - \left[a \left(x_n - \frac{1}{a}\right)\right]^2. \end{aligned}$$

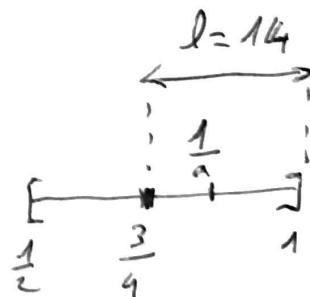
5. $e_{n+1} = -e_n^2 = -[-e_{n-1}]^2 = -e_{n-1}^4$ etc...

$$\rightarrow |e_n| = |e_0|^{2^n}$$

La méthode va converger si $|e_0| < 1$

Cela suggère de prendre de manière optimale :

$$x_0 = \frac{3}{4} \Rightarrow |e_0| \leq \frac{1}{4}$$



$$\begin{aligned} \rightarrow |e_n| &= \frac{1}{4^{2^n}} < \epsilon_{tol} \rightarrow n > \ln_2 \left[\frac{\ln(\epsilon_{tol})}{\ln(1/4)} \right] \\ \epsilon_{tol} &= 2^{-52} \rightarrow \boxed{n=5} \end{aligned}$$