Exercice A.1.29 Gradient et Laplacien en coordonnées polaires

1. Avec les règles de dérivation de fonctions composées (vues dans les exercices **13.** et **14.**,) on a le système suivant

$$\begin{cases} \frac{\partial g}{\partial r}(r,\theta) &= \cos\theta\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+\sin\theta\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta) \quad L_1 \\ \frac{\partial g}{\partial \theta}(r,\theta) &= -r\sin\theta\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)+r\cos\theta\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)\ . \quad L_2 \end{cases}$$

Dans la suite, on doit supposer que $r \neq 0$.

2. Maintenant on résoud ce système d'inconnues $\frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta)$ et $\frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta)$.

$$\cos\theta L_1 - \frac{\sin\theta}{r}L_2 \Rightarrow \frac{\partial f}{\partial x}(r\cos\theta,r\sin\theta) = (\cos\theta)\frac{\partial g}{\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}(r,\theta) = g_1(r,\theta)$$

$$\sin\theta L_1 + \frac{\cos\theta}{r}L_2 \Rightarrow \frac{\partial f}{\partial y}(r\cos\theta,r\sin\theta) = (\sin\theta)\frac{\partial g}{\partial r}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}(r,\theta) = g_2(r,\theta)$$

On obtient ainsi l'expression du gradient en coordonnées polaires vues en PS21 :

$$\boxed{\nabla f(r\cos\theta,r\sin\theta) = \frac{\partial g}{\partial r}(r,\theta)\vec{e}_r + \frac{1}{r}\frac{\partial g}{\partial\theta}(r,\theta)\vec{e}_\theta} \quad , \text{ où } \vec{e}_r = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \text{ et } \vec{e}_\theta = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

3. On a

$$\frac{\partial g_1}{\partial r}(r,\theta) = (\cos\theta)\frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{\sin\theta}{r^2}\frac{\partial g}{\partial\theta}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}(r,\theta)$$

$$\frac{\partial g_1}{\partial\theta}(r,\theta) = -(\sin\theta)\frac{\partial g}{\partial r}(r,\theta) + (\cos\theta)\frac{\partial^2 g}{\partial\theta\partial r}(r,\theta) - \frac{\cos\theta}{r}\frac{\partial g}{\partial\theta}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial^2 g}{\partial\theta^2}(r,\theta)$$

$$\frac{\partial g_2}{\partial r}(r,\theta) = (\sin\theta)\frac{\partial^2 g}{\partial r^2}(r,\theta) - \frac{\cos\theta}{r^2}\frac{\partial g}{\partial\theta}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}(r,\theta)$$

$$\frac{\partial g_2}{\partial\theta}(r,\theta) = (\cos\theta)\frac{\partial g}{\partial r}(r,\theta) + (\sin\theta)\frac{\partial^2 g}{\partial\theta\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial g}{\partial\theta}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial^2 g}{\partial\theta^2}(r,\theta)$$

4. D'après ce qui précède sachant que $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ et en utilisant la symétrie de Schwarz, on peut écrire

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(r\cos\theta,r\sin\theta) &= (\cos\theta)\frac{\partial g_1}{\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial g_1}{\partial\theta}(r,\theta)\\ &= (\cos\theta)^2\frac{\partial^2 g}{\partial r^2}(r,\theta) + 2\frac{\cos\theta\sin\theta}{r^2}\frac{\partial g}{\partial\theta}(r,\theta) - 2\frac{\cos\theta\sin\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}(r,\theta)\\ &\quad + \frac{(\sin\theta)^2}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{(\sin\theta)^2}{r^2}\frac{\partial^2 g}{\partial\theta^2}(r,\theta).\end{aligned}$$

Sachant que $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ et en utilisant la symétrie de Schwarz, on peut écrire

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2}(r\cos\theta,r\sin\theta) &= (\sin\theta)\frac{\partial g_2}{\partial r}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial g_2}{\partial\theta}(r,\theta)\\ &= (\sin\theta)^2\frac{\partial^2 g}{\partial r^2}(r,\theta) - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial g}{\partial\theta}(r,\theta) + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 g}{\partial r\partial\theta}(r,\theta)\\ &\quad + \frac{(\cos\theta)^2}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{(\cos\theta)^2}{r^2}\frac{\partial^2 g}{\partial\theta^2}(r,\theta).\end{aligned}$$

5. Pour $r \neq 0$, le Laplacien de f en coordonnées polaires est la somme des deux dernières expressions :

$$\begin{aligned}(\Delta f)(r\cos\theta, r\sin\theta) &= \frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) + \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\\ &= (\cos\theta)^2\frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{(\sin\theta)^2}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{(\sin\theta)^2}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)\\ &\quad + (\sin\theta)^2\frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{(\cos\theta)^2}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{(\cos\theta)^2}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)\end{aligned}$$

$$\boxed{(\Delta f)(r\cos\theta, r\sin\theta) = \frac{\partial^2 g}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial g}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}(r,\theta)}$$

Exercice A.2.1. Opérateurs différentiels
Dans tout l'exercice on pose

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, (fA) = \begin{pmatrix} fA_1 \\ fA_2 \\ fA_3 \end{pmatrix}$$

$$\overrightarrow{\Delta} A = \begin{pmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \end{pmatrix} \quad \text{et} \quad (B \cdot \boldsymbol{\nabla})A = \begin{pmatrix} B \cdot \boldsymbol{\nabla} A_1 \\ B \cdot \boldsymbol{\nabla} A_2 \\ B \cdot \boldsymbol{\nabla} A_3 \end{pmatrix} = \begin{pmatrix} B_1 \frac{\partial A_1}{\partial x} + B_2 \frac{\partial A_1}{\partial y} + B_3 \frac{\partial A_1}{\partial z} \\ B_1 \frac{\partial A_2}{\partial x} + B_2 \frac{\partial A_2}{\partial y} + B_3 \frac{\partial A_2}{\partial z} \\ B_1 \frac{\partial A_3}{\partial x} + B_2 \frac{\partial A_3}{\partial y} + B_3 \frac{\partial A_3}{\partial z} \end{pmatrix}$$

0. $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$

$$A \wedge (B \wedge C) = A \wedge \begin{pmatrix} B_2C_3 - B_3C_2 \\ B_3C_1 - B_1C_3 \\ B_1C_2 - B_2C_1 \end{pmatrix} = \begin{pmatrix} A_2(B_1C_2 - B_2C_1) - A_3(B_3C_1 - B_1C_3) \\ A_3(B_2C_3 - B_3C_2) - A_1(B_1C_2 - B_2C_1) \\ A_1(B_3C_1 - B_1C_3) - A_2(B_2C_3 - B_3C_2) \end{pmatrix}$$

$$(A \cdot C)B = \begin{pmatrix} B_1(A_1C_1 + A_2C_2 + A_3C_3) \\ B_2(A_1C_1 + A_2C_2 + A_3C_3) \\ B_3(A_1C_1 + A_2C_2 + A_3C_3) \end{pmatrix} \quad \text{et} \quad (A \cdot B)C = \begin{pmatrix} C_1(A_1B_1 + A_2B_2 + A_3B_3) \\ C_2(A_1B_1 + A_2B_2 + A_3B_3) \\ C_3(A_1B_1 + A_2B_2 + A_3B_3) \end{pmatrix}$$

$$\Rightarrow (A \cdot C)B - (A \cdot B)C = \begin{pmatrix} B_1(A_2C_2 + A_3C_3) - C_1(A_2B_2 + A_3B_3) \\ B_2(A_1C_1 + A_3C_3) - C_2(A_1B_1 + A_3B_3) \\ B_3(A_1C_1 + A_2C_2) - C_3(A_1B_1 + A_2B_2) \end{pmatrix} = A \wedge (B \wedge C)$$

3. $\operatorname{div}(\boldsymbol{\nabla} f) = \Delta f$ → vu en cours.
4. $\operatorname{div}(f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\Delta g - g\Delta f$
Ici il faut utiliser les questions **2.** et **3.** et la commutativité du produit scalaire

$$\begin{aligned} \operatorname{div}(f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = \operatorname{div}(f\boldsymbol{\nabla} g) - \operatorname{div}(g\boldsymbol{\nabla} f) &\underset{\mathbf{2.}}{=} \Big(f \operatorname{div}(\boldsymbol{\nabla} g) + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g\Big) - \Big(g \operatorname{div}(\boldsymbol{\nabla} f) + \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f\Big) \\ &= f \operatorname{div}(\boldsymbol{\nabla} g) - g \operatorname{div}(\boldsymbol{\nabla} f) \underset{\mathbf{3.}}{=} f\Delta g - g\Delta f \end{aligned}$$

6. $\operatorname{div}(A \wedge B) = B \cdot \mathbf{rot}\, A - A \cdot \mathbf{rot}\, B$

$$A \wedge B = \begin{pmatrix} A_2B_3 - A_3B_2 \\ A_3B_1 - A_1B_3 \\ A_1B_2 - A_2B_1 \end{pmatrix} \Rightarrow \operatorname{div}(A \wedge B) = \frac{\partial(A_2B_3 - A_3B_2)}{\partial x} + \frac{\partial(A_3B_1 - A_1B_3)}{\partial y} + \frac{\partial(A_1B_2 - A_2B_1)}{\partial z}$$

$$\begin{aligned} \operatorname{div}(A \wedge B) &= \boxed{\frac{\partial A_2}{\partial x} \times B_3} + A_2 \times \frac{\partial B_3}{\partial x} - \boxed{\frac{\partial A_3}{\partial x} \times B_2} - A_3 \frac{\partial B_2}{\partial x} \\ &+ \boxed{\frac{\partial A_3}{\partial y} \times B_1} + A_3 \times \frac{\partial B_1}{\partial y} - \boxed{\frac{\partial A_1}{\partial y} \times B_3} - A_1 \times \frac{\partial B_3}{\partial y} \\ &+ \boxed{\frac{\partial A_1}{\partial z} \times B_2} + A_1 \times \frac{\partial B_2}{\partial z} - \boxed{\frac{\partial A_2}{\partial z} \times B_1} - A_2 \times \frac{\partial B_1}{\partial z} \\ &= \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \\ \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} \\ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \end{pmatrix} \\ &= B \cdot \mathbf{rot}\, A - A \cdot \mathbf{rot}\, B \end{aligned}$$

7. div **rot** $A = 0 \to$ vu en cours.

9. $\mathbf{rot}(A \wedge B) = A\,\mathrm{div}\,B - B\,\mathrm{div}\,A + (B\cdot\boldsymbol{\nabla})A - (A\cdot\boldsymbol{\nabla})B$

$$
\begin{aligned}
\mathbf{rot}(A\wedge B) &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} A_2B_3 - A_3B_2 \\ A_3B_1 - A_1B_3 \\ A_1B_2 - A_2B_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial(A_1B_2 - A_2B_1)}{\partial y} - \frac{\partial(A_3B_1 - A_1B_3)}{\partial z} \\ \frac{\partial(A_2B_3 - A_3B_2)}{\partial z} - \frac{\partial(A_1B_2 - A_2B_1)}{\partial x} \\ \frac{\partial(A_3B_1 - A_1B_3)}{\partial x} - \frac{\partial(A_2B_3 - A_3B_2)}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial A_1}{\partial y}\times B_2 + \boxed{A_1 \times \frac{\partial B_2}{\partial y} - \frac{\partial A_2}{\partial y}\times B_1} - A_2 \times \frac{\partial B_1}{\partial y} - \left(\boxed{\frac{\partial A_3}{\partial z}\times B_1} + A_3 \times \frac{\partial B_1}{\partial z} - \frac{\partial A_1}{\partial z}\times B_3 - \boxed{A_1 \times \frac{\partial B_3}{\partial z}}\right) \\ \frac{\partial A_2}{\partial z}\times B_3 + \boxed{A_2 \times \frac{\partial B_3}{\partial z} - \frac{\partial A_3}{\partial z}\times B_2} - A_3 \times \frac{\partial B_2}{\partial z} - \left(\boxed{\frac{\partial A_1}{\partial x}\times B_2} + A_1 \times \frac{\partial B_2}{\partial x} - \frac{\partial A_2}{\partial x}\times B_1 - \boxed{A_2 \times \frac{\partial B_1}{\partial x}}\right) \\ \frac{\partial A_3}{\partial x}\times B_1 + \boxed{A_3 \times \frac{\partial B_1}{\partial x} - \frac{\partial A_1}{\partial x}\times B_3} - A_1 \times \frac{\partial B_2}{\partial x} - \left(\boxed{\frac{\partial A_2}{\partial y}\times B_3} + A_2 \times \frac{\partial B_3}{\partial y} - \frac{\partial A_3}{\partial y}\times B_2 - \boxed{A_3 \times \frac{\partial B_2}{\partial y}}\right) \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} A_1\left(\frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} + \frac{\partial B_1}{\partial x}\right) \\ A_2\left(\frac{\partial B_3}{\partial z} + \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y}\right) \\ A_3\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) \end{pmatrix}}_{=A(\mathrm{div}\,B)} - \underbrace{\begin{pmatrix} B_1\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x}\right) \\ B_2\left(\frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y}\right) \\ B_3\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \end{pmatrix}}_{=B(\mathrm{div}\,A)} \\
&\qquad + \begin{pmatrix} B_2\frac{\partial A_1}{\partial y} - A_2\frac{\partial B_1}{\partial y} - \left(A_3\frac{\partial B_1}{\partial z} - B_3\frac{\partial A_1}{\partial z}\right) - A_1\frac{\partial B_1}{\partial x} + B_1\frac{\partial A_1}{\partial x} \\ B_3\frac{\partial A_2}{\partial z} - A_3\frac{\partial B_2}{\partial z} - \left(A_1\frac{\partial B_2}{\partial x} - B_1\frac{\partial A_2}{\partial x}\right) - A_2\frac{\partial B_2}{\partial y} + B_2\frac{\partial A_2}{\partial y} \\ B_1\frac{\partial A_3}{\partial x} - A_1\frac{\partial B_2}{\partial x} - \left(A_2\frac{\partial B_3}{\partial y} - B_2\frac{\partial A_3}{\partial y}\right) - A_3\frac{\partial B_3}{\partial z} + B_3\frac{\partial A_3}{\partial z} \end{pmatrix}
\end{aligned}
$$

Étudions la dernière matrice :

$$
\begin{aligned}
&\begin{pmatrix} B_2\frac{\partial A_1}{\partial y} - A_2\frac{\partial B_1}{\partial y} - \left(A_3\frac{\partial B_1}{\partial z} - B_3\frac{\partial A_1}{\partial z}\right) - A_1\frac{\partial B_1}{\partial x} + B_1\frac{\partial A_1}{\partial x} \\ B_3\frac{\partial A_2}{\partial z} - A_3\frac{\partial B_2}{\partial z} - \left(A_1\frac{\partial B_2}{\partial x} - B_1\frac{\partial A_2}{\partial x}\right) - A_2\frac{\partial B_2}{\partial y} + B_2\frac{\partial A_2}{\partial y} \\ B_1\frac{\partial A_3}{\partial x} - A_1\frac{\partial B_2}{\partial x} - \left(A_2\frac{\partial B_3}{\partial y} - B_2\frac{\partial A_3}{\partial y}\right) - A_3\frac{\partial B_3}{\partial z} + B_3\frac{\partial A_3}{\partial z} \end{pmatrix} \\
&\quad = \underbrace{\begin{pmatrix} B_1\frac{\partial A_1}{\partial x} + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z} \\ B_3\frac{\partial A_2}{\partial z} + B_2\frac{\partial A_2}{\partial y} + B_1\frac{\partial A_2}{\partial x} \\ B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} + B_3\frac{\partial A_3}{\partial z} \end{pmatrix}}_{=(B\cdot\boldsymbol{\nabla})A} - \underbrace{\begin{pmatrix} A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z} \\ A_3\frac{\partial B_2}{\partial z} + A_2\frac{\partial B_2}{\partial y} + A_1\frac{\partial B_2}{\partial x} \\ A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z} \end{pmatrix}}_{(A\cdot\boldsymbol{\nabla})B}
\end{aligned}
$$

10. $\boldsymbol{\nabla}(A\cdot B) = A\wedge \mathbf{rot}\, B + B\wedge \mathbf{rot}\, A + (B\cdot\boldsymbol{\nabla})A + (A\cdot\boldsymbol{\nabla})B$. D'après la question **1.**, on a

$$\boldsymbol{\nabla}(A\cdot B) = \boldsymbol{\nabla}(A_1B_1+A_2B_2+A_3B_3) = B_1(\boldsymbol{\nabla}A_1)+A_1(\boldsymbol{\nabla}B_1)+B_2(\boldsymbol{\nabla}A_2)+A_2(\boldsymbol{\nabla}B_2)+B_3(\boldsymbol{\nabla}A_3)+A_3(\boldsymbol{\nabla}B_3)$$

Ensuite on calcule,

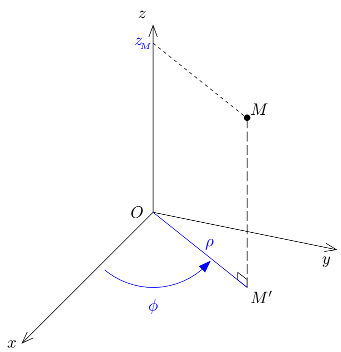
$$\begin{aligned}
A\wedge \mathbf{rot}\, B + (A\cdot\boldsymbol{\nabla})B &= \begin{pmatrix} A_1\\ A_2\\ A_3\end{pmatrix} \wedge \begin{pmatrix} \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\\ \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\\ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\end{pmatrix} + \begin{pmatrix} A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\\ A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_2}{\partial y} + A_3\frac{\partial B_2}{\partial z}\\ A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z}\end{pmatrix}\\
&= \begin{pmatrix} A_2(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}) - A_3(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x})\\ A_3(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}) - A_1(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y})\\ A_1(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}) - A_2(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z})\end{pmatrix} + \begin{pmatrix} A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\\ A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_2}{\partial y} + A_3\frac{\partial B_2}{\partial z}\\ A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z}\end{pmatrix}\\
&= \begin{pmatrix} A_2\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_3}{\partial x} + \boxed{A_1\frac{\partial B_1}{\partial x}}\\ A_3\frac{\partial B_3}{\partial y} + \boxed{A_1\frac{\partial B_1}{\partial y}} + A_2\frac{\partial B_2}{\partial y}\\ \boxed{A_1\frac{\partial B_1}{\partial z}} + A_2\frac{\partial B_2}{\partial z} + A_3\frac{\partial B_3}{\partial z}\end{pmatrix} = A_1\boldsymbol{\nabla}B_1 + A_2\boldsymbol{\nabla}B_2 + A_3\boldsymbol{\nabla}B_3
\end{aligned}$$

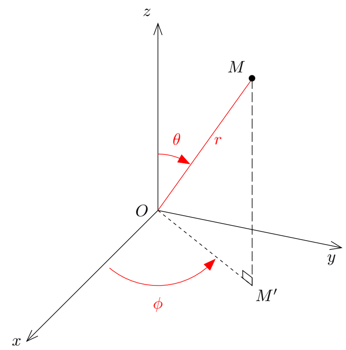
Par symétrie on a également (en échangeant les rôles de A et B)

$$B\wedge \mathbf{rot}\, A + (B\cdot\boldsymbol{\nabla})A = B_1\boldsymbol{\nabla}A_1 + B_2\boldsymbol{\nabla}A_2 + B_3\boldsymbol{\nabla}A_3$$

On ajoutant les quatres termes, on obtient bien l'égalité escomptée.

Exercice A.2.3. Question 1
Soit $M(x_M, y_M, z_M)$ un point de l'espace $(Oxyz)$.





Les coordonnées cylindriques de M sont (ρ, ϕ, z_M)

$$\begin{cases} x_M &= OM' \cos\phi = \rho\cos\phi \\ y_M &= OM' \sin\phi = \rho\sin\phi \\ z_M \end{cases}$$

$\rho \in [0, +\infty[$, $\phi \in [0, 2\pi[$ et $z_M \in \mathbb{R}$

Les coordonnées sphériques de M sont (r, ϕ, θ)

$$\begin{cases} x_M &= OM' \cos\phi = r\cos\phi\sin\theta \\ y_M &= OM' \sin\phi = r\sin\phi\sin\theta \\ z_M &= MM' = r\cos\theta \end{cases}$$

$r \in [0, +\infty[$, $\phi \in [0, 2\pi[$ et $\theta \in [0, \pi]$.

1. (a) Nous avons trouvé $\Delta f(x,y) = \frac{1}{(x^2+y^2)^{\frac{3}{2}}}$ et $\Delta g(x,y,z) = 0$.
(b) Il faut utiliser les formules du Laplacien disponibles dans le poly dans différents systèmes de coordonnées. Elles ne sont pas à apprendre par cœur !

Système de coordonnées polaires : si on pose $\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta) = \frac{1}{r}$, alors

$$\Delta f(r\cos\theta, r\sin\theta) = \frac{\partial^2 \tilde{f}}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial \tilde{f}}{\partial r}(r,\theta) + \underbrace{\ldots}_{=0} = \left(\frac{2}{r^3}\right) + \frac{1}{r} \times \left(-\frac{1}{r^2}\right) = \frac{1}{r^3}.$$

Système de coordonnées cylindriques : Si on pose $\tilde{g}(\rho,\phi,z) = g(\rho\cos\phi, \rho\sin\phi, z) = \frac{1}{\sqrt{\rho^2+z^2}}$ alors

$$\Delta g(\rho\cos\phi, \rho\sin\phi, z) = \frac{\partial^2 \tilde{g}}{\partial \rho^2}(\rho,\phi,z) + \frac{1}{\rho}\frac{\partial \tilde{g}}{\partial \rho}(\rho,\phi,z) + \underbrace{\ldots}_{=0} + \frac{\partial^2 \tilde{g}}{\partial z^2}(\rho,\phi,z)$$

Or, $\tilde{g}(\rho,\theta,\phi) = f(\rho, z)$ donc on utilise les résultat du (a) avec $x = \rho$ et $y = z$.

$$\Delta g(\rho\cos\phi, \rho\sin\phi, z) = \frac{3\rho^2 - r^2}{r^5} + \frac{1}{\rho} \times \left(-\frac{\rho}{r^3}\right) + \frac{3z^2 - r^2}{r^5} = \frac{[3\rho^2 - r^2] - [r^2] + [3z^2 - r^2]}{r^5} = 0$$

où on rappelle que $r^2 = x^2 + y^2 + z^2$.

Système de coordonnées sphériques : Si on pose $\tilde{g}(r,\theta,\phi) = g(r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta) = \frac{1}{r}$ alors

$$\Delta g(r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta) = \frac{\partial^2 \tilde{g}}{\partial r^2}(r,\theta,\phi) + \frac{2}{\rho}\frac{\partial \tilde{g}}{\partial r}(r,\theta,\phi) + \underbrace{\ldots}_{=0} = \left(\frac{2}{r^3}\right) + \frac{2}{r} \times \left(-\frac{1}{r^2}\right) = 0.$$

Tous les calculs mènent aux mêmes résultats, quelque soit le système de coordonnées utilisé !

Exercice A.2.6. Potentiel scalaire
1. On considère le champ de vecteurs suivant

$$\begin{array}{rcl} V:\mathbb{R}^3 & \to & \mathbb{R}^3 \\ M & \mapsto & V(M)=\begin{pmatrix} -\dfrac{xz}{r^3} \\ \dfrac{-yz}{r^3} \\ \dfrac{1}{r}-\dfrac{z^2}{r^3} \end{pmatrix}. \end{array}$$

Les composantes du champs de vecteurs V admettent des dérivées partielles continues sur $\mathbb{R}^3\backslash\{0\}$. Le champ de vecteurs V dérive d'un potentiel scalaire ssi $\mathbf{rot}\, V=\vec{0}$. On calcule

$$\mathbf{rot}\, V(M)=\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}\times\begin{pmatrix} -\dfrac{xz}{r^3} \\ \dfrac{-yz}{r^3} \\ \dfrac{1}{r}-\dfrac{z^2}{r^3} \end{pmatrix}=\begin{pmatrix} -\dfrac{y}{r^3}+\dfrac{3z^2y}{r^5}+\dfrac{y}{r^3}-\dfrac{3yz^2}{r^5} \\ -\dfrac{x}{r^3}+\dfrac{3xz^2}{r^5}+\dfrac{x}{r^3}-\dfrac{3xz^2}{r^5} \\ \dfrac{3yzx}{r^5}-\dfrac{3xyz}{r^5} \end{pmatrix}=\vec{0}$$

On en déduit que V dérive d'un potentiel scalaire f.
Maintenant, on détermine le champ scalaire f tel que $V=\nabla f$. On commence par poser le système : On a

$$\left\{\begin{array}{rcll} \dfrac{\partial f}{\partial x}(x,y,z) & = & -\dfrac{xz}{r^3} & (1) \\ \dfrac{\partial f}{\partial y}(x,y,z) & = & -\dfrac{yz}{r^3} & (2) \\ \dfrac{\partial f}{\partial z}(x,y,z) & = & \dfrac{1}{r}-\dfrac{z^2}{r^3} & (3) \end{array}\right.$$

Etape 1 : On intègre (1) par rapport à la variable x

$$(4)\qquad f(x,y,z)=\int -\frac{xz}{r^3}dx=z\int -\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}dx=z\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}+C_1(y,z)=\frac{z}{r}+C_1(y,z)$$

Etape 2 : On dérive (4) par rapport à la variable y et on identifie l'expression à (2)

$$\frac{\partial f}{\partial y}(x,y,z)=-z\frac{1}{2}\frac{2y}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{\partial C_1}{\partial y}(y,z)=-\frac{yz}{r}+\frac{\partial C_1}{\partial y}(y,z)$$

On en déduit que

$$\frac{\partial C_1}{\partial y}(y,z)=0.$$

On intègre cette équation par rapport à la variable y et on trouve

$$C_1(y,z)=C_2(z)$$

A la fin de l'étape 2 on a

$$(5)\qquad f(x,y,z)=\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}+C_2(z)$$

Etape 3 : On dérive (5) par rapport à la variable z et on identifie l'expression à (3)

$$\frac{\partial f}{\partial z}(x,y,z)=\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}-z\frac{1}{2}\frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{\partial C_2}{\partial z}(z)=\frac{1}{r}-\frac{z^2}{r^3}+\frac{\partial C_2}{\partial z}(z)$$

On en déduit que

$$\frac{\partial C_2}{\partial z}(z) = 0.$$

On intègre cette équation par rapport à la variable z et on trouve

$$C_2(z) = C, \quad C \in \mathbb{R}$$

Conclusion :

$$f(x, y, z) = \frac{z}{r} + C, \quad C \in \mathbb{R} \, .$$

2. Ici, il faut comprendre que le champ V est de la forme

$$V(M) = \begin{pmatrix} 2xz\phi(z) \\ -2yz\phi(z) \\ -(x^2 - y^2)\phi(z) \end{pmatrix}$$

Si on veut que $\mathbf{rot}\, V = \vec{0}$ alors ϕ est solution de l'équation différentielle homogène suivante

$$z\phi'(z) + 2\phi(z) = 0 \, .$$

D'après ce que vous avez vu en MT91, on résoud l'équation sur $]-\infty; 0[\cup]0 \; +\infty[$ et vous savez que

$$\phi(z) = C \exp(\int -\frac{2}{z} dz) = C \exp(-2 \ln z) = \frac{C}{z^2}, \quad C \in \mathbb{R} \, .$$

Pour la suite c'est pareil qu'au **1.** On pose le système $V = \nabla f$ (en remlaçant ϕ par son expression :

$$\left\{ \begin{array}{rcll} \dfrac{\partial f}{\partial x}(x, y, z) & = & 2C\dfrac{x}{z} & (1) \\ \dfrac{\partial f}{\partial y}(x, y, z) & = & -2C\dfrac{y}{z} & (2) \\ \dfrac{\partial f}{\partial z}(x, y, z) & = & -C\dfrac{x^2 - y^2}{z^2} & (3) \end{array} \right.$$

Etape 1 : On intègre (1) par rapport à la variable x

$$f(x, y, z) = C\frac{1}{z} \int 2x \, dx = C\frac{x^2}{z} + C_1(y, z) \qquad (4)$$

Etape 2 : On dérive (4) par rapport à la variable y et on identifie l'expression à (2)

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial C_1}{\partial y}(y, z)$$

On en déduit que

$$\frac{\partial C_1}{\partial y}(y, z) = -2C\frac{y}{z}.$$

On intègre cette équation par rapport à la variable y et on trouve

$$C_1(y, z) = -C\frac{y^2}{z} + C_2(z)$$

A la fin de l'étape 2 on a

$$f(x, y, z) = C\frac{x^2 - y^2}{z} + C_2(z) \qquad (5)$$

Etape 3 : On dérive (5) par rapport à la variable z et on identifie l'expression à (3)

$$\frac{\partial f}{\partial z}(x,y,z) = -C\frac{x^2-y^2}{z^2} + \frac{\partial C_2}{\partial z}(z)$$

On en déduit que

$$\frac{\partial C_2}{\partial z}(z) = 0.$$

On intègre cette équation par rapport à la variable z et on trouve

$$C_2(z) = K, \quad C \in \mathbb{R}$$

Conclusion :

$$f(x,y,z) = C\frac{x^2-y^2}{z} + K, \quad C, K \in \mathbb{R}\ .$$

3. Plus précisément on a $W = \begin{pmatrix} \frac{x}{r^3} \\ \frac{y}{r^3} \\ \frac{z}{r^3} \end{pmatrix}$.

On trouve $W = \nabla f$ avec $f(x,y,z) = -\frac{1}{r} + C, \quad C \in \mathbb{R}$.