Exercice A.1.29 Gradient et Laplacien en coordonnées polaires

1. Avec les règles de dérivation de fonctions composées (vues dans les exercices **13.** et **14.**,) on a le système suivant

$$\left\{\begin{array}{lcl} \frac{\partial g}{\partial r}(r,\theta) & = & \cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + \sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \qquad L_1 \\ \frac{\partial g}{\partial \theta}(r,\theta) & = & -r\sin\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) + r\cos\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\ . \quad L_2 \end{array}\right.$$

Dans la suite, on doit supposer que $r \neq 0$.

2. Maintenant on résoud ce système d'inconnues $\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)$ et $\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)$.

$$\cos\theta L_1 - \frac{\sin\theta}{r}L_2 \Rightarrow \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) = (\cos\theta)\frac{\partial g}{\partial r}(r,\theta) - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}(r,\theta) = g_1(r,\theta)$$

$$\sin\theta L_1 + \frac{\cos\theta}{r}L_2 \Rightarrow \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) = (\sin\theta)\frac{\partial g}{\partial r}(r,\theta) + \frac{\cos\theta}{r}\frac{\partial g}{\partial \theta}(r,\theta) = g_2(r,\theta)$$

On obtient ainsi l'expression du gradient en coordonnées polaires vues en PS21 :

$$\boxed{\nabla f(r\cos\theta, r\sin\theta) = \frac{\partial g}{\partial r}(r,\theta)\vec{e}_r + \frac{1}{r}\frac{\partial g}{\partial \theta}(r,\theta)\vec{e}_\theta} \quad ,\ \text{où } \vec{e}_r = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \text{ et } \vec{e}_\theta = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

Exercice A.2.1. Opérateurs différentiels
Dans tout l'exercice on pose

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, (fA) = \begin{pmatrix} fA_1 \\ fA_2 \\ fA_3 \end{pmatrix}$$

$$\overrightarrow{\Delta} A = \begin{pmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \end{pmatrix} \quad \text{et} \quad (B \cdot \boldsymbol{\nabla})A = \begin{pmatrix} B \cdot \boldsymbol{\nabla} A_1 \\ B \cdot \boldsymbol{\nabla} A_2 \\ B \cdot \boldsymbol{\nabla} A_3 \end{pmatrix} = \begin{pmatrix} B_1 \frac{\partial A_1}{\partial x} + B_2 \frac{\partial A_1}{\partial y} + B_3 \frac{\partial A_1}{\partial z} \\ B_1 \frac{\partial A_2}{\partial x} + B_2 \frac{\partial A_2}{\partial y} + B_3 \frac{\partial A_2}{\partial z} \\ B_1 \frac{\partial A_3}{\partial x} + B_2 \frac{\partial A_3}{\partial y} + B_3 \frac{\partial A_3}{\partial z} \end{pmatrix}$$

1. $\boldsymbol{\nabla}(fg) = g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g.$

$$\nabla(fg) = \begin{pmatrix} \frac{\partial(fg)}{\partial x} \\ \frac{\partial(fg)}{\partial y} \\ \frac{\partial(fg)}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \times g + f \times \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} \times g + f \times \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} \times g + f \times \frac{\partial g}{\partial z} \end{pmatrix} = g \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} + f \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} = g\boldsymbol{\nabla} f + f\boldsymbol{\nabla} g$$

2. $\mathrm{div}(fA) = f \,\mathrm{div}\, A + A \cdot \boldsymbol{\nabla} f$

$$\begin{aligned} \mathrm{div}(fA) &= \frac{\partial(fA_1)}{\partial x} + \frac{\partial(fA_2)}{\partial y} + \frac{\partial(fA_3)}{\partial z} \\ &= \frac{\partial f}{\partial x} \times A_1 + \boxed{f \times \frac{\partial A_1}{\partial x}} + \frac{\partial f}{\partial y} \times A_2 + \boxed{f \times \frac{\partial A_2}{\partial y}} + \frac{\partial f}{\partial z} \times A_3 + \boxed{f \times \frac{\partial A_3}{\partial z}} \\ &= f\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(\frac{\partial f}{\partial x} \times A_1 + \frac{\partial f}{\partial y} \times A_2 + \frac{\partial f}{\partial z} \times A_3\right) \\ &= f \,\mathrm{div}\, A + \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = f \,\mathrm{div}\, A + A \cdot \boldsymbol{\nabla} f \end{aligned}$$

3. $\mathrm{div}(\boldsymbol{\nabla} f) = \Delta f$ $\rightarrow$ vu en cours.
5. $\mathbf{rot}(fA) = f\,\mathbf{rot}\, A + \boldsymbol{\nabla} f \wedge A$

$$\mathbf{rot}(fA) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} fA_1 \\ fA_2 \\ fA_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial(fA_3)}{\partial y} - \frac{\partial(fA_2)}{\partial z} \\ \frac{\partial(fA_1)}{\partial z} - \frac{\partial(fA_3)}{\partial x} \\ \frac{\partial(fA_2)}{\partial x} - \frac{\partial(fA_1)}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial y} \times A_3 + f \times \frac{\partial A_3}{\partial y} - \left(\frac{\partial f}{\partial z} \times A_2 + f \times \frac{\partial A_2}{\partial z}\right) \\ \frac{\partial f}{\partial z} \times A_1 + f \times \frac{\partial A_1}{\partial z} - \left(\frac{\partial f}{\partial x} \times A_3 + f \times \frac{\partial A_3}{\partial x}\right) \\ \frac{\partial f}{\partial x} \times A_2 + f \times \frac{\partial A_2}{\partial x} - \left(\frac{\partial f}{\partial y} \times A_1 + f \times \frac{\partial A_1}{\partial y}\right) \end{pmatrix}$$

$$= f \underbrace{\begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix}}_{= \mathbf{rot}\, A} + \underbrace{\begin{pmatrix} \frac{\partial f}{\partial y} \times A_3 - \frac{\partial f}{\partial z} \times A_2 \\ \frac{\partial f}{\partial z} \times A_1 - \frac{\partial f}{\partial x} \times A_3 \\ \frac{\partial f}{\partial x} \times A_2 - \frac{\partial f}{\partial y} \times A_1 \end{pmatrix}}_{= \boldsymbol{\nabla} f \wedge A}$$

8. $\mathbf{rot}(\mathbf{rot}\,A) = \boldsymbol{\nabla}\,\mathrm{div}\,A - \overrightarrow{\Delta} A$

$$\mathbf{rot}(\mathbf{rot}\,A) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}) - \frac{\partial}{\partial z}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) \\ \frac{\partial}{\partial z}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) - \frac{\partial}{\partial x}(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}) \\ \frac{\partial}{\partial x}(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}) - \frac{\partial}{\partial y}(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^2 A_2}{\partial y \partial x} - \boxed{\frac{\partial^2 A_1}{\partial y^2}}) - (\boxed{\frac{\partial^2 A_1}{\partial z^2}} - \frac{\partial^2 A_3}{\partial z \partial x}) \\ (\frac{\partial^2 A_3}{\partial z \partial y} - \boxed{\frac{\partial^2 A_2}{\partial z^2}}) - (\boxed{\frac{\partial^2 A_2}{\partial x^2}} - \frac{\partial^2 A_1}{\partial x \partial y}) \\ (\frac{\partial^2 A_1}{\partial x \partial z} - \boxed{\frac{\partial^2 A_3}{\partial x^2}}) - (\boxed{\frac{\partial^2 A_3}{\partial y^2}} - \frac{\partial^2 A_2}{\partial y \partial z}) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} - \frac{\partial^2 A_1}{\partial x^2} \\ -\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} \\ -\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial Z^2} \end{pmatrix}}_{= -\overrightarrow{\Delta} A} + \underbrace{\begin{pmatrix} +\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \\ +\frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial y} + \frac{\partial^2 A_1}{\partial x \partial y} \\ +\frac{\partial^2 A_3}{\partial z^2} + \frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} \end{pmatrix}}_{= \boldsymbol{\nabla}(\mathrm{div}\,A)}$$

On conclut la dernière égalité en supposant A de classe $\mathscr{C}^2$ pour utiliser le théorème de symétrie de Schwarz.

Exercice A.2.3. Question 2
Pour appliquer les résultats précédents, posons

$$f(x,y,z) = g(r) = g(\sqrt{x^2+y^2+z^2}) \text{ et } \vec{A}(x,y,z) = \overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

On a

$$\operatorname{div}(f\vec{A}) = f \operatorname{div}\vec{A} + \nabla f \cdot \vec{A} \quad \Leftrightarrow \quad g(r)\operatorname{div}\overrightarrow{OM} + g'(r)\frac{\overrightarrow{OM}}{r}\cdot\overrightarrow{OM} = 0.$$

En dimension 2, on applique A.1.29 Q2 à $f(M) = g(r)$, on a bien

$$\nabla f(M) = \frac{\partial g}{\partial r}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \underbrace{\frac{\partial g}{\partial \theta}}_{=0}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{\partial g}{\partial r}\begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \end{pmatrix} = \frac{\partial g}{\partial r}\frac{\overrightarrow{OM}}{r}.$$

Le résultat reste vrai en dimension 3.
Sachant que

$$\operatorname{div}\overrightarrow{OM} = 3 \qquad \text{et} \qquad \overrightarrow{OM}\cdot\overrightarrow{OM} = x^2+y^2+z^2 = r^2,$$

on obtient on équation différentielle homogène du 1er ordre pour la fonction g :

$$\boxed{3g(r) + rg'(r) = 0}.$$

On rappelle la formule vue en MT02 : la forme générale des solutions de $y' = a(t)y$ est $y(t) = Ce^{\int a(t)dt}$ où C se détermine à l'aide d'une condition initiale.
On a donc

$$\boxed{g'(r) = -\frac{3}{r}g(r) \Rightarrow g(r) = Ce^{-3\ln r} = Ce^{\ln r^{-3}} = \frac{C}{r^3}}.$$

Comme $\|\nabla f(-1,1,0)\| = \left\|g'(\sqrt{2})\right\|$ car $\|\overrightarrow{OM}\| = r \Rightarrow \frac{\overrightarrow{OM}}{r} = 1$, la condition intiale se réécrit

$$|g'(\sqrt{2})| = 5 \Leftrightarrow \frac{3}{\sqrt{2}} \times \frac{C}{(\sqrt{2})^3} = 5 \Leftrightarrow \boxed{C = \pm\frac{20}{3}}.$$

Exercice A.2.5. Potentiel scalaire
À faire !