

Exercice A.2.1. Opérateurs différentiels

Dans tout l'exercice on pose

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, (fA) = \begin{pmatrix} fA_1 \\ fA_2 \\ fA_3 \end{pmatrix}$$

$$\vec{\Delta} A = \begin{pmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \end{pmatrix} \quad \text{et} \quad (B \cdot \nabla) A = \begin{pmatrix} B \cdot \nabla A_1 \\ B \cdot \nabla A_2 \\ B \cdot \nabla A_3 \end{pmatrix} = \begin{pmatrix} B_1 \frac{\partial A_1}{\partial x} + B_2 \frac{\partial A_1}{\partial y} + B_3 \frac{\partial A_1}{\partial z} \\ B_1 \frac{\partial A_2}{\partial x} + B_2 \frac{\partial A_2}{\partial y} + B_3 \frac{\partial A_2}{\partial z} \\ B_1 \frac{\partial A_3}{\partial x} + B_2 \frac{\partial A_3}{\partial y} + B_3 \frac{\partial A_3}{\partial z} \end{pmatrix}$$

0. $A \wedge (B \wedge C) = (A \cdot C)B - (A \cdot B)C$

$$A \wedge (B \wedge C) = A \wedge \begin{pmatrix} B_2 C_3 - B_3 C_2 \\ B_3 C_1 - B_1 C_3 \\ B_1 C_2 - B_2 C_1 \end{pmatrix} = \begin{pmatrix} A_2(B_1 C_2 - B_2 C_1) - A_3(B_3 C_1 - B_1 C_3) \\ A_3(B_2 C_3 - B_3 C_2) - A_1(B_1 C_2 - B_2 C_1) \\ A_1(B_3 C_1 - B_1 C_3) - A_2(B_2 C_3 - B_3 C_2) \end{pmatrix}$$

$$(A \cdot C)B = \begin{pmatrix} B_1(A_1 C_1 + A_2 C_2 + A_3 C_3) \\ B_2(A_1 C_1 + A_2 C_2 + A_3 C_3) \\ B_3(A_1 C_1 + A_2 C_2 + A_3 C_3) \end{pmatrix} \quad \text{et} \quad (A \cdot B)C = \begin{pmatrix} C_1(A_1 B_1 + A_2 B_2 + A_3 B_3) \\ C_2(A_1 B_1 + A_2 B_2 + A_3 B_3) \\ C_3(A_1 B_1 + A_2 B_2 + A_3 B_3) \end{pmatrix}$$

$$\Rightarrow (A \cdot C)B - (A \cdot B)C = \begin{pmatrix} B_1(A_2 C_2 + A_3 C_3) - C_1(A_2 B_2 + A_3 B_3) \\ B_2(A_1 C_1 + A_3 C_3) - C_2(A_1 B_1 + A_3 B_3) \\ B_3(A_1 C_1 + A_2 C_2) - C_3(A_1 B_1 + A_2 B_2) \end{pmatrix} = A \wedge (B \wedge C)$$

4. $\text{div}(f \nabla g - g \nabla f) = f \Delta g - g \Delta f$

Ici il faut utiliser les questions 2. et 3. et la commutativité du produit scalaire

$$\begin{aligned} \text{div}(f \nabla g - g \nabla f) &= \text{div}(f \nabla g) - \text{div}(g \nabla f) \stackrel{2.}{=} \left(f \text{div}(\nabla g) + \nabla f \cdot \nabla g \right) - \left(g \text{div}(\nabla f) + \nabla g \cdot \nabla f \right) \\ &= f \text{div}(\nabla g) - g \text{div}(\nabla f) \stackrel{3.}{=} f \Delta g - g \Delta f \end{aligned}$$

6. $\text{div}(A \wedge B) = B \cdot \text{rot } A - A \cdot \text{rot } B$

$$A \wedge B = \begin{pmatrix} A_2 B_3 - A_3 B_2 \\ A_3 B_1 - A_1 B_3 \\ A_1 B_2 - A_2 B_1 \end{pmatrix} \Rightarrow \text{div}(A \wedge B) = \frac{\partial(A_2 B_3 - A_3 B_2)}{\partial x} + \frac{\partial(A_3 B_1 - A_1 B_3)}{\partial y} + \frac{\partial(A_1 B_2 - A_2 B_1)}{\partial z}$$

$$\begin{aligned} \text{div}(A \wedge B) &= \boxed{\frac{\partial A_2}{\partial x} \times B_3} + A_2 \times \frac{\partial B_3}{\partial x} - \boxed{\frac{\partial A_3}{\partial x} \times B_2} - A_3 \frac{\partial B_2}{\partial x} \\ &\quad + \boxed{\frac{\partial A_3}{\partial y} \times B_1} + A_3 \times \frac{\partial B_1}{\partial y} - \boxed{\frac{\partial A_1}{\partial y} \times B_3} - A_1 \times \frac{\partial B_3}{\partial y} \\ &\quad + \boxed{\frac{\partial A_1}{\partial z} \times B_2} + A_1 \times \frac{\partial B_2}{\partial z} - \boxed{\frac{\partial A_2}{\partial z} \times B_1} - A_2 \times \frac{\partial B_1}{\partial z} \\ &= \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \\ \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} \\ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \end{pmatrix} \\ &= B \cdot \text{rot } A - A \cdot \text{rot } B \end{aligned}$$

7. $\text{div rot } A = 0 \rightarrow$ **vu en cours.**

9. $\text{rot}(A \wedge B) = A \text{ div } B - B \text{ div } A + (B \cdot \nabla)A - (A \cdot \nabla)B$

$$\begin{aligned}
\text{rot}(A \wedge B) &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} A_2 B_3 - A_3 B_2 \\ A_3 B_1 - A_1 B_3 \\ A_1 B_2 - A_2 B_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial(A_1 B_2 - A_2 B_1)}{\partial y} - \frac{\partial(A_3 B_1 - A_1 B_3)}{\partial z} \\ \frac{\partial(A_2 B_3 - A_3 B_2)}{\partial z} - \frac{\partial(A_1 B_2 - A_2 B_1)}{\partial x} \\ \frac{\partial(A_3 B_1 - A_1 B_3)}{\partial x} - \frac{\partial(A_2 B_3 - A_3 B_2)}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial A_1}{\partial y} \times B_2 + \boxed{A_1 \times \frac{\partial B_2}{\partial y} - \frac{\partial A_2}{\partial y} \times B_1} - A_2 \times \frac{\partial B_1}{\partial y} - \left(\boxed{\frac{\partial A_3}{\partial z} \times B_1} + A_3 \times \frac{\partial B_1}{\partial z} - \frac{\partial A_1}{\partial z} \times B_3 - \boxed{A_1 \times \frac{\partial B_3}{\partial z}} \right) \\ \frac{\partial A_2}{\partial z} \times B_3 + \boxed{A_2 \times \frac{\partial B_3}{\partial z} - \frac{\partial A_3}{\partial z} \times B_2} - A_3 \times \frac{\partial B_2}{\partial z} - \left(\boxed{\frac{\partial A_1}{\partial x} \times B_2} + A_1 \times \frac{\partial B_2}{\partial x} - \frac{\partial A_2}{\partial x} \times B_1 - \boxed{A_2 \times \frac{\partial B_1}{\partial x}} \right) \\ \frac{\partial A_3}{\partial x} \times B_1 + \boxed{A_3 \times \frac{\partial B_1}{\partial x} - \frac{\partial A_1}{\partial x} \times B_3} - A_1 \times \frac{\partial B_2}{\partial x} - \left(\boxed{\frac{\partial A_2}{\partial y} \times B_3} + A_2 \times \frac{\partial B_3}{\partial y} - \frac{\partial A_3}{\partial y} \times B_2 - \boxed{A_3 \times \frac{\partial B_2}{\partial y}} \right) \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} A_1 \left(\frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} + \frac{\partial B_1}{\partial x} \right) \\ A_2 \left(\frac{\partial B_3}{\partial z} + \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} \right) \\ A_3 \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) \end{pmatrix}}_{= A(\text{div } B)} - \underbrace{\begin{pmatrix} B_1 \left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x} \right) \\ B_2 \left(\frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} \right) \\ B_3 \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \end{pmatrix}}_{= B(\text{div } A)} \\
&\quad + \begin{pmatrix} B_2 \frac{\partial A_1}{\partial y} - A_2 \frac{\partial B_1}{\partial y} - \left(A_3 \frac{\partial B_1}{\partial z} - B_3 \frac{\partial A_1}{\partial z} \right) - \frac{\partial A_1}{\partial x} B_1 + \frac{\partial A_1}{\partial x} B_1 \\ B_3 \frac{\partial A_2}{\partial z} - A_3 \frac{\partial B_2}{\partial z} - \left(A_1 \frac{\partial B_2}{\partial x} - B_1 \frac{\partial A_2}{\partial x} \right) - \frac{\partial A_2}{\partial y} B_2 + \frac{\partial A_2}{\partial y} B_2 \\ B_1 \frac{\partial A_3}{\partial x} - A_1 \frac{\partial B_2}{\partial x} - \left(A_2 \frac{\partial B_3}{\partial y} - B_2 \frac{\partial A_3}{\partial y} \right) - \frac{\partial A_3}{\partial z} B_3 + \frac{\partial A_3}{\partial z} B_3 \end{pmatrix}
\end{aligned}$$

Étudions la dernière matrice :

$$\begin{aligned}
&\begin{pmatrix} B_2 \frac{\partial A_1}{\partial y} - A_2 \frac{\partial B_1}{\partial y} - \left(A_3 \frac{\partial B_1}{\partial z} - B_3 \frac{\partial A_1}{\partial z} \right) - \frac{\partial A_1}{\partial x} B_1 + \frac{\partial A_1}{\partial x} B_1 \\ B_3 \frac{\partial A_2}{\partial z} - A_3 \frac{\partial B_2}{\partial z} - \left(A_1 \frac{\partial B_2}{\partial x} - B_1 \frac{\partial A_2}{\partial x} \right) - \frac{\partial A_2}{\partial y} B_2 + \frac{\partial A_2}{\partial y} B_2 \\ B_1 \frac{\partial A_3}{\partial x} - A_1 \frac{\partial B_2}{\partial x} - \left(A_2 \frac{\partial B_3}{\partial y} - B_2 \frac{\partial A_3}{\partial y} \right) - \frac{\partial A_3}{\partial z} B_3 + \frac{\partial A_3}{\partial z} B_3 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \frac{\partial A_1}{\partial x} B_1 + B_2 \frac{\partial A_1}{\partial y} + B_3 \frac{\partial A_1}{\partial z} \\ B_3 \frac{\partial A_2}{\partial z} + \frac{\partial A_2}{\partial y} B_2 + B_1 \frac{\partial A_2}{\partial x} \\ B_1 \frac{\partial A_3}{\partial x} + B_2 \frac{\partial A_3}{\partial y} + \frac{\partial A_3}{\partial z} B_3 \end{pmatrix}}_{=(B \cdot \nabla)A} - \underbrace{\begin{pmatrix} \frac{\partial A_1}{\partial x} B_1 + A_2 \frac{\partial B_1}{\partial y} + A_3 \frac{\partial B_1}{\partial z} \\ A_3 \frac{\partial B_2}{\partial z} + \frac{\partial B_2}{\partial y} A_2 + A_1 \frac{\partial B_2}{\partial x} \\ A_1 \frac{\partial B_2}{\partial x} + A_2 \frac{\partial B_3}{\partial y} + \frac{\partial B_3}{\partial z} A_3 \end{pmatrix}}_{=(A \cdot \nabla)B}
\end{aligned}$$

10. $\nabla(A \cdot B) = A \wedge \mathbf{rot} B + B \wedge \mathbf{rot} A + (B \cdot \nabla)A + (A \cdot \nabla)B$ D'après la question 1., on a

$$\nabla(A \cdot B) = \nabla(A_1 B_1 + A_2 B_2 + A_3 B_3) = B_1(\nabla A_1) + A_1(\nabla B_1) + B_2(\nabla A_2) + A_2(\nabla B_2) + B_3(\nabla A_3) + A_3(\nabla B_3)$$

Ensuite on calcule,

$$\begin{aligned} A \wedge \mathbf{rot} B + (A \cdot \nabla)B &= \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \\ \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} \\ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \end{pmatrix} + \begin{pmatrix} A_1 \frac{\partial B_1}{\partial x} + A_2 \frac{\partial B_1}{\partial y} + A_3 \frac{\partial B_1}{\partial z} \\ A_1 \frac{\partial B_2}{\partial x} + A_2 \frac{\partial B_2}{\partial y} + A_3 \frac{\partial B_2}{\partial z} \\ A_1 \frac{\partial B_3}{\partial x} + A_2 \frac{\partial B_3}{\partial y} + A_3 \frac{\partial B_3}{\partial z} \end{pmatrix} \\ &= \begin{pmatrix} A_2(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}) - A_3(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}) \\ A_3(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}) - A_1(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}) \\ A_1(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}) - A_2(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}) \end{pmatrix} + \begin{pmatrix} A_1 \frac{\partial B_1}{\partial x} + A_2 \frac{\partial B_1}{\partial y} + A_3 \frac{\partial B_1}{\partial z} \\ A_1 \frac{\partial B_2}{\partial x} + A_2 \frac{\partial B_2}{\partial y} + A_3 \frac{\partial B_2}{\partial z} \\ A_1 \frac{\partial B_3}{\partial x} + A_2 \frac{\partial B_3}{\partial y} + A_3 \frac{\partial B_3}{\partial z} \end{pmatrix} \\ &= \begin{pmatrix} A_2 \frac{\partial B_2}{\partial x} + A_3 \frac{\partial B_3}{\partial x} + \boxed{A_1 \frac{\partial B_1}{\partial x}} \\ A_3 \frac{\partial B_3}{\partial y} + \boxed{A_1 \frac{\partial B_1}{\partial y}} + A_2 \frac{\partial B_2}{\partial y} \\ \boxed{A_1 \frac{\partial B_1}{\partial z}} + A_2 \frac{\partial B_2}{\partial z} + A_3 \frac{\partial B_3}{\partial z} \end{pmatrix} = A_1 \nabla B_1 + A_2 \nabla B_2 + A_3 \nabla B_3 \end{aligned}$$

Par symétrie on a également (en échangeant les rôles de A et B)

$$B \wedge \mathbf{rot} A + (B \cdot \nabla)A = B_1 \nabla A_1 + B_2 \nabla A_2 + B_3 \nabla A_3$$

On ajoutant les quatres termes, on obtient bien l'égalité escomptée.

Exercice A.2.6. Potentiel scalaire

1. On considère le champ de vecteurs suivant

$$V : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$M \mapsto V(M) = \begin{pmatrix} -\frac{xz}{r^3} \\ -\frac{yz}{r^3} \\ \frac{1}{r} - \frac{z^2}{r^3} \end{pmatrix}.$$

Les composantes du champs de vecteurs V admettent des dérivées partielles continues sur $\mathbb{R}^3 \setminus \{0\}$. Le champ de vecteurs V dérive d'un potentiel scalaire ssi $\text{rot } V = \vec{0}$. On calcule

$$\text{rot } V(M) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} -\frac{xz}{r^3} \\ -\frac{yz}{r^3} \\ \frac{1}{r} - \frac{z^2}{r^3} \end{pmatrix} = \begin{pmatrix} -\frac{y}{r^3} + \frac{3z^2y}{r^5} + \frac{y}{r^3} - \frac{3yz^2}{r^5} \\ -\frac{x}{r^3} + \frac{3xz^2}{r^5} + \frac{x}{r^3} - \frac{3xz^2}{r^5} \\ \frac{3yzx}{r^5} - \frac{3xyz}{r^5} \end{pmatrix} = \vec{0}$$

On en déduit que V dérive d'un potentiel scalaire f .

Maintenant, on détermine le champ scalaire f tel que $V = \nabla f$. On commence par poser le système : On a

$$\begin{cases} \frac{\partial f}{\partial x}(x, y, z) = -\frac{xz}{r^3} & (1) \\ \frac{\partial f}{\partial y}(x, y, z) = -\frac{yz}{r^3} & (2) \\ \frac{\partial f}{\partial z}(x, y, z) = \frac{1}{r} - \frac{z^2}{r^3} & (3) \end{cases}$$

Etape 1 : On intègre (1) par rapport à la variable x

$$(4) \quad f(x, y, z) = \int -\frac{xz}{r^3} dx = z \int -\frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx = z \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + C_1(y, z) = \frac{z}{r} + C_1(y, z)$$

Etape 2 : On dérive (4) par rapport à la variable y et on identifie l'expression à (2)

$$\frac{\partial f}{\partial y}(x, y, z) = -z \frac{1}{2} \frac{2y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial C_1}{\partial y}(y, z) = -\frac{yz}{r} + \frac{\partial C_1}{\partial y}(y, z)$$

On en déduit que

$$\frac{\partial C_1}{\partial y}(y, z) = 0.$$

On intègre cette équation par rapport à la variable y et on trouve

$$C_1(y, z) = C_2(z)$$

A la fin de l'étape 2 on a

$$(5) \quad f(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + C_2(z)$$

Etape 3 : On dérive (5) par rapport à la variable z et on identifie l'expression à (3)

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} - z \frac{1}{2} \frac{2z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + \frac{\partial C_2}{\partial z}(z) = \frac{1}{r} - \frac{z^2}{r^3} + \frac{\partial C_2}{\partial z}(z)$$

On en déduit que

$$\frac{\partial C_2}{\partial z}(z) = 0.$$

On intègre cette équation par rapport à la variable z et on trouve

$$C_2(z) = C, \quad C \in \mathbb{R}$$

Conclusion :

$$f(x, y, z) = \frac{z}{r} + C, \quad C \in \mathbb{R}.$$

2. Ici, il faut comprendre que le champ V est de la forme

$$V(M) = \begin{pmatrix} 2xz\phi(z) \\ -2yz\phi(z) \\ -(x^2 - y^2)\phi(z) \end{pmatrix}$$

Si on veut que $\mathbf{rot} V = \vec{0}$ alors ϕ est solution de l'équation différentielle homogène suivante

$$z\phi'(z) + 2\phi(z) = 0.$$

D'après ce que vous avez vu en MT91, on résout l'équation sur $] -\infty; 0[\cup]0 +\infty[$ et vous savez que

$$\phi(z) = C \exp\left(\int -\frac{2}{z} dz\right) = C \exp(-2 \ln z) = \frac{C}{z^2}, \quad C \in \mathbb{R}.$$

Pour la suite c'est pareil qu'au **1**. On pose le système $V = \nabla f$ (en remplaçant ϕ par son expression :

$$\begin{cases} \frac{\partial f}{\partial x}(x, y, z) = 2C \frac{x}{z} & (1) \\ \frac{\partial f}{\partial y}(x, y, z) = -2C \frac{y}{z} & (2) \\ \frac{\partial f}{\partial z}(x, y, z) = -C \frac{x^2 - y^2}{z^2} & (3) \end{cases}$$

Etape 1 : On intègre (1) par rapport à la variable x

$$(4) \quad f(x, y, z) = C \frac{1}{z} \int 2x dx = C \frac{x^2}{z} + C_1(y, z)$$

Etape 2 : On dérive (4) par rapport à la variable y et on identifie l'expression à (2)

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial C_1}{\partial y}(y, z)$$

On en déduit que

$$\frac{\partial C_1}{\partial y}(y, z) = -2C \frac{y}{z}.$$

On intègre cette équation par rapport à la variable y et on trouve

$$C_1(y, z) = -C \frac{y^2}{z} + C_2(z)$$

A la fin de l'étape 2 on a

$$(5) \quad f(x, y, z) = C \frac{x^2 - y^2}{z} + C_2(z)$$

Etape 3 : On dérive (5) par rapport à la variable z et on identifie l'expression à (3)

$$\frac{\partial f}{\partial z}(x, y, z) = -C \frac{x^2 - y^2}{z^2} + \frac{\partial C_2}{\partial z}(z)$$

On en déduit que

$$\frac{\partial C_2}{\partial z}(z) = 0.$$

On intègre cette équation par rapport à la variable z et on trouve

$$C_2(z) = K, \quad C \in \mathbb{R}$$

Conclusion :

$$f(x, y, z) = C \frac{x^2 - y^2}{z} + K, \quad C, K \in \mathbb{R}.$$

3. Plus précisément on a $W = \begin{pmatrix} \frac{x}{r^3} \\ \frac{y}{r^3} \\ \frac{z}{r^3} \end{pmatrix}$.

On trouve $W = \nabla f$ avec $f(x, y, z) = -\frac{1}{r} + C, \quad C \in \mathbb{R}.$