Decision under uncertainty: formal probabilities

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Questions (partially) answered in this lecture

- How to model our uncertainty about X?
  - **by probabilities (decision under risk)**
  - by sets (decision under uncertainty)
  - by more general models
- How to take decisions?
  - **by considering complete rankings through expected utility**
  - by allowing incomparabilities

Outline

- Expected utility
  - Utilities
  - Probabilities and expected utility
  - Why this framework?

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Classical model

- Each consequence c associated to utility $u(c)$
- Each action/state a/x pair has a utility of the consequence
- Matrix $\mathcal{U}$ becomes $u(a, x)$ values

	x_1	x_2	$\ldots$	x_n
a_1		$u(a_1, x_2)$		
a_2				$u(a_2, x_n)$
$\vdots$		$\ddots$		
a_m				

An example

We want to cross a sea stretch:

- States: sea weather conditions
- Actions: type of transports

States $\mathscr{X}$

x_1 =Calm sea x_2 =Agitated sea x_3 =Stormy weather



Actions $\mathscr{A}$

a_1 =Motor boat a_2 =Catamaran a_3 =Ferry boat



The matrix $\mathscr{U}$

	x_1	x_2	x_3
a_1	12	0	−10
a_2	−2	8	0
a_3	1	5	10

Which action to choose?

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Basic definitions

Basic tool

A probability distribution $p : \mathscr{X} \to [0,1]$ such that

- $p(x) \geq 0$
- $\sum_x p(x) = 1$

from which for any subset we have

- $P(A) = \sum_{x \in A} p(x)$
- $P(A) = 1 - P(A^c)$: auto-dual

Example (Academic dice)

Assume a dice, we have $\mathscr{X} = \{1, 2, \ldots, 6\}$:

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = {}^1/_6$$

$$P(\{1,3,5\}) = {}^1/_6 + {}^1/_6 + {}^1/_6 = {}^1/_2$$

Definition

Given the values $u(a,x)$, the action a and a probability p over states, expected utility defined as

$$\mathbb{E}(a) = \sum_x u(a,x)p(x)$$

is used to evaluate a. An action a_k is consider better than a_ℓ if

$$\mathbb{E}(a_k) \geq \mathbb{E}(a_\ell),$$

that is we can expect a higher reward for a_k.

An example

Storm is unlikely ($p(x_3) = 0.1$), most probably agitated ($p(x_2) = 0.7$) but maybe calm ($p(x_1) = 0.2$)

	x_1	x_2	x_3	$\mathbb{E}(a_i)$
a_1	12	0	−10	1.4
a_2	−2	8	0	
a_3	1	5	10	
Max				

$$\mathbb{E}(a_1) = 0.2 \cdot 12 + 0.7 \cdot 0 + 0.1 \cdot -10 = 1.4$$

An example

Storm is unlikely ($p(x_3) = 0.1$), most probably agitated ($p(x_2) = 0.7$) but maybe calm ($p(x_1) = 0.2$)

	x_1	x_2	x_3	$\mathbb{E}(a_i)$
a_1	12	0	−10	1.4
a_2	−2	8	0	5.2
a_3	1	5	10	4.7
Max				5.2

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Can you provide everyday life examples where it is used?

Would you use it in every situations involving uncertainty?

Why utilities, expectation and probability?

Many different justifications, but the main ones are:

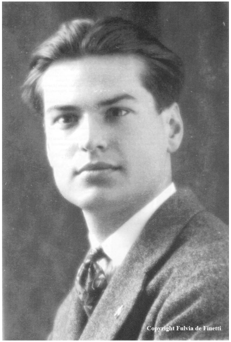
- **Von Neumann-Morgenstern** [3] assumes probabilities are given, derive utilities and expectation from 4 axioms
- **De Finetti** [1] assumes utilities are given, derive probability and expectation from 2 axioms
- **Savage** [2] assumes neither, derive utilities, probabilities and expectations from 7 qualitative axioms (4 central + 3 more technical)

$\Rightarrow$ one common point in all axiomatics is to assume that **any two acts are comparable** $\rightarrow$ completeness of preferences

The guys: one for each possible next lecture

L. Savage

J. Von Neumann



B. De Finetti

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