Decision under uncertainty: the savage approach

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Why probabilities?

Can you provide everyday life examples where probabilities are used?

Would you use it in every situations involving uncertainty?

→ This lecture: a justification to the answer "yes", according to Savage

Why utilities, expectation and probability?

In this talk, we will consider

- **Von Neumann-Morgenstern** [3] assumes probabilities are given, derive utilities and expectation from 4 axioms
- **De Finetti** [1] assumes utilities are given, derive probability and expectation from 2 axioms
- **Savage** [2] that assumes neither, derive utilities, probabilities and expectations from 7 qualitative axioms (4 central + 3 more technical)

⇒ one common point in all axiomatics is to assume that **any two acts are comparable** → completeness of preferences

The guys

L. **Savage**

J. Von Neumann

B. De Finetti

The basics

- The set of possible states $\mathscr{X}$ of the problem
- The set of possible consequences $\mathscr{C}$
- An act a is a function that associate to each state a consequence

$$a : \mathscr{X} \to \mathscr{C}$$

	$\nearrow\nearrow$	$\nearrow$	$\sim$
a_1: buy	\$\$\$	\$\$	0
a_2: negotiate	\$\$	\$\$	\$
a_3: do nothing	\$	\$	\$

- If an act a is at least as good as b (to you), we denote it $a \geq b$

Total preferences

Axiom 1 (A1)

The preference relation among acts $\leq$ is a total pre-order

From this it follows:

- That any pair of acts a, b can be compared: either $a \geq b$, $b \geq a$, or both (indifference)
- That there are worst $\underline{a}$ and best $\overline{a}$ acts, with $\underline{a} \leq a \leq \overline{a}$ for any a
- That all consequences in $\mathscr{C}$ are totally ordered (just consider constant acts)

Formal sure-thing principle

Your preferences only depend on states for which consequences differ

Axiom 2 (A2)

- a, b coincide on set B of states, and $a \leq b$
- a', b' coincide with a, b on $\neg B$ and between them on B
- we can conclude $a' \leq b'$

From this it follows that we can speak of a preferred to b given B ($a \leq_B b$), that is when we look only at consequences for B

Sure-thing principle example

Consider choosing an outing, where you are unsure whether

- It will be with friends (F) or without ($\neg F$)
- It will be raining (R) or not ($\neg R$)

You can choose one of the four outings

- a : Bowling
- b : Movie theater
- a' : Picnic
- b' : Cycling

Sure-thing principle in action

	$R \wedge F$	$R \wedge \neg F$	$\neg R \wedge \neg F$	$\neg R \wedge F$
a: Cycling	☺	☺	☺	☺☺☺
b: Movie	☺☺☺	☺	☺	☺
a′: Picnic	☺	∼	∼	☺☺☺
b′: Bowl	☺☺☺	∼	∼	☺

- If *Cycling* ≥ *Movie*, then
- *Picnic* ≥ *Bowling*

A bit of notation for the rest

Given two acts a, b, we denote by $a_E b$ the act such that

$$a_E b(s) = \begin{cases} a(s) & \text{if } s \in E \\ b(s) & \text{else} \end{cases}$$

An example on $a_R b$

	$R \wedge F$	$R \wedge \neg F$	$\neg R \wedge \neg F$	$\neg R \wedge F$
a: Cycling	☺	☺	☺	☺☺☺
b: Movie	☺☺☺	☺	☺	☺
$a_R b$	☺	☺	☺	☺
$b_R a$	☺☺☺	☺	☺	☺☺☺

Sure-thing principle restated

Axiom 2 revisited

For all acts a, b, a', b' and every event E

$$a_E b \geq a'_E b \text{ if and only if } a_E b' \geq a'_E b'$$

More economical/elegant, perhaps a bit less intuitive.

Ordinal Event Independence

Replacing a consequence by a better one on a set of states can only increase an act attractiveness

Axiom 3 (A3)

For every (non-null) event E and consequences c, d

$$c \geq d \text{ if and only if } c_E a \geq d_E a$$

	$R \wedge F$	$R \wedge \neg F$	$\neg R \wedge \neg F$	$\neg R \wedge F$
a: Cycling	☺	☺	☺	☺☺☺
b: Movie	☺☺☺	☺	☺	☺
a': Picnic	☺	~	~	☺☺☺
b': Bowl	☺☺☺	~	~	☺

Cycling $\geq$ Picnic, because identical on F, constant on $\neg F$

Comparative probabilities

Comparison between events likelihood should not depend on how important are the consequences difference

Axiom 4 (A4)

For every events E, F and every consequences c, d, c', d' with

$$c \geq d \text{ and } c' \geq d'$$

then

$$c_E d \geq c_{E'} d \text{ if and only if } c'_E d' \geq c'_{E'} d'$$

Basically, if E is more likely than E', it is always so

Illustration

	$R \wedge F$	$R \wedge \neg F$	$\neg R \wedge \neg F$	$\neg R \wedge F$
a : ☺☺$_R$☺	☺☺	☺☺	☺	☺
b : ☺☺$_F$☺	☺☺	☺	☺	☺☺
a' : ☺☺☺$_R$ ~	☺☺☺	☺☺☺	~	~
b' : ☺☺☺$_F$ ~	☺☺☺	~	~	☺☺☺

- Under Axiom 4, preferences between a, b same as between a', b'
- ☺☺$_R$☺ $\geq$ ☺☺$_F$☺ can then be interpreted as "R more likely than F"

The Savage wrap-up

The main result

- Given states $\mathscr{X}$ and consequences in $\mathscr{C}$
- Given that preferences $\geq$ on act satisfy axioms $A1 - A4$ [a]
- Then, $\geq$ can be uniquely represented by probability p on $\mathscr{X}$ + utility[b] u on consequences such that

$$a \geq b$$

if and only if

$$\sum_x p(x)u(a(x)) \geq \sum_x p(x)u(b(x))$$

[a] + couple of more technical ones $A5 - A7$ to ensure non-degeneracy and handle unbounded utilities/acts

[b] Up to positive linear scaling → we can take any $u' = a \cdot u + b$ with $a > 0$

What does it tell us?

- If you are facing a choice with uncertain consequences
- If you accept axioms $A1 - A4$ as **acceptable/rational**

then, you **should** act/decide **as if** there was some probabilities and utilities in your head[1].

The theory also tells you that

- those can be derived from observing your preferences
- in particular, they do not need to come from frequencies[2], they can be personal (and depending on your experience).

[1] No claim there are, just that all happens "as if" → valid as reasoning models/tools
[2] But they can if it makes sense

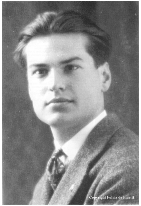
Next lecture

Three options:

- Go see what one of the other guys has to say?

L. Savage

J. Von Neumann



B. De Finetti

- Probabilities and expected utilities are great, cool! But are they perfect? Let's go see some critics of them (and possibly what we can do about it).
- I like probabilities but do not trust utilities, can I do something about it? Let's see some other ways to compare probabilities

References I

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[2] L. Savage.
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[3] J. von Neumann and O. Morgenstern.
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