Decision under uncertainty: de Finetti and betting

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Why probabilities?

Can you provide everyday life examples where probabilities are used?

Would you use it in every situations involving uncertainty?

→ This lecture: a justification to the answer "yes", according to de Finetti

Why utilities, expectation and probability?

In this talk, we will consider

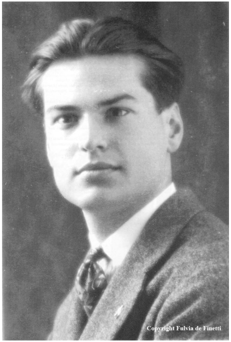
- **Von Neumann-Morgenstern** [5] assumes probabilities are given, derive utilities and expectation from 4 axioms
- **De Finetti** [3] assumes utilities are given, derive probability and expectation from 2 axioms
- **Savage** [4] that assumes neither, derive utilities, probabilities and expectations from 7 qualitative axioms (4 central + 3 more technical)

⇒ one common point in all axiomatics is to assume that **any two acts are comparable** → completeness of preferences

The guys

L. Savage

J. Von Neumann



B. De Finetti

DeFinetti and gambles/acts

- Assume that linear utilities are already given, i.e., an act/gamble

$$a : \mathscr{X} \to \mathbb{R}$$

  is a function with $a(x)$ the rewards when x happens;
- The price $\mathbb{E}(a)$ you consider fair to buy/sell this gamble for reflect your belief ($\sim$ probabilities) about the world
- Give an operational means to measure probabilities

An example of bet/gamble

Assume that in a distant future, Alpha-Omega ($\alpha\omega$), the almost-but-not-quite sentient AI, is going to play 3-D Go against the Augmented champion Lee Se-dol the 13th. The betting droid proposes you the following gamble in credits ₢, with $\mathscr{X} = \{\alpha\omega \; \textit{Wins}, \textit{Loses}, \textit{Ties}\}$:

$$a(W) = 1000₢, a(L) = -500₢, a(T) = -200₢$$

what is the price $\mathbb{E}(a)$, in credit, you would associate to it?

Combining bets and coherence

How can we define a coherent agent (You)?

- You receive n gambles $a_1,\ldots,a_n$;
- You give prices $\mathbb{E}(a_1),\ldots,\mathbb{E}(a_n)$;
- The gain of a ticket, if x happens, is $a(x)-\mathbb{E}(a)$
- c_i is the number of tickets we make you buy ($c_i>0$) or sell ($c_i<0$) of a_i at the given price.

you are rational if there is no choice of c_i's such that

$$\sup_x \sum_{i=1}^{n} c_i[a(x)-\mathbb{E}(a)]<0,$$

which means you lose whatever happens (even in best-case scenario).

- if bets are coherent, prices $\mathbb{E}(a_i)$ are expectation of a probability

Axioms characterizing a coherent bet (only 2!)

- Additivity: if a, b are two gambles of prices $\mathbb{E}(a), \mathbb{E}(b)$, then

$$\mathbb{E}(a+b) = \mathbb{E}(a) + \mathbb{E}(b).$$

  You are ready to pay $\mathbb{E}(a) + \mathbb{E}(b)$ to play to a and b

- Boundedness:

$$\inf_x a(x) \leq \mathbb{E}(a) \leq \sup_x a(x).$$

  You would not buy for more than you can win ($\leq \sup_x a(x)$), nor sell for less than you can lose ($\inf_x a(x) \leq$).

- Linearity (as a consequence): if $\alpha > 0$ and μ constant then

$$\mathbb{E}(\alpha a + \mu) = \alpha \mathbb{E}(a) + \mu$$

The case of events

- If gamble takes only values $\{0, 1\}$, then it is an indicator $\mathbb{I}_E$ of some event E:

$$\mathbb{I}_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ else} \end{cases}$$

- Price $\mathbb{E}(\mathbb{I}_E) = P(E)$ interpreted as the (subjective) probability of E
- This means that the gain from a ticket to bet on an event is

$$\mathbb{I}_E(x) - P(E)$$

A simple example

- You bet on raining ($X = R$) or not raining ($X = \neg R$)
- You specify the following prices:

$$P(R) = 0.7, \quad P(\neg R) = 0.5$$

- Show that you can specify numbers $c_R, c_{\neg R}$ of tickets bought/sold such that you are guaranteed to lose.

Additivity of probabilities

- the elements $\mathscr{X} = \{x_1, \ldots, x_n\}$
- let $p(x_i)$ be the prices of gamble on x_i
- the sum $\sum_{i=1}^{N} \mathbb{I}_{x_i}(x) = 1$ is constant s(one and only one x_i happens)
- if $\sum_{i=1}^{N} p(x_i) > 1$, buy one ticket each

$$\sum_{i=1}^{N} \mathbb{I}_{x_i}(x) - p(x_i) = 1 - \sum_{i=1}^{N} p(x_i) < 0$$

- if $\sum_{i=1}^{N} p(x_i) > 1$, sell one ticket each

$$\sum_{i=1}^{N} p(x_i) - \mathbb{I}_{x_i}(x) = \sum_{i=1}^{N} p(x_i) - 1 < 0$$

- Hence, to not lose money for sure, **we must have** $\sum_{i=1}^{N} p(x_i) = 1$

Remember our example?

Example (Facing tomorrow's market)

As head of a company selling AI.stuff, you know that the market demand is unlikely to **go crazy** ($\nearrow\nearrow$), will most probably **go up reasonably** ($\nearrow$), but is also unlikely to **remain stable** ($\sim$). You can either:

- Buy a new machine (a_1)
  - If market stable ($\sim$), loss of money (0)
  - If market increase, higher income ($$)
  - If market go crazy, you become rich ($$$)
- Negotiate extra hours with employees if needed (a_2)
  - If market stable ($\sim$), normal income ($)
  - If market increase or go crazy, higher income ($$)
- Do nothing about it (a_3)
  - Whatever happens, normal income ($)

How would you model this decision problem? What would you recommend?

Probability of crazy market?

	$\nearrow\nearrow$	$\nearrow$	$\sim$
a	1000	0	0

- How much your financial analyst would bet on it?
- The more he/she want to bet, the more it is likely
- Since utilities are defined up to affine positive scaling,

$$P(\{\nearrow\nearrow\}) = \mathbb{E}(a)/1000$$

- To see that, consider that

$$\mathbb{E}(a) = 1000 \cdot P(\{\nearrow\nearrow\}) + 0 \cdot P(\{\nearrow\}) + 0 \cdot P(\{\sim\})$$

Probabilities revisited

Main interpretations

- **Frequentist :** $P(A)$= number of times A observed in a population
  - only apply when frequencies make sense
  - what if I want to describe uncertainty on a one-time event?
  - on a quantity that is fixed but ill-known?
- **Subjectivist :** $P(A)$= price for gamble giving 1 if A happens, 0 if not
  - applies to any kind of quantities (even those occuring once or being fixed)
  - e.g.: allows you to attach a probability to the statement "The next united state president will be a woman"

Next lecture

Three options:

- Go see what one of the other guys has to say?

L. Savage

J. Von Neumann

B. De Finetti

- Probabilities and expected utilities are great, cool! But are they perfect? Let's go see some critics of them (and possibly what we can do about it).
- I like probabilities but do not trust utilities, can I do something about it? Let's see some other ways to compare probabilities

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