Decision under uncertainty: de Finetti and betting

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Why probabilities?

Can you provide everyday life examples where probabilities are used?

Would you use it in every situations involving uncertainty?

→ This lecture: a justification to the answer "yes", according to de Finetti

Why utilities, expectation and probability?

In this talk, we will consider

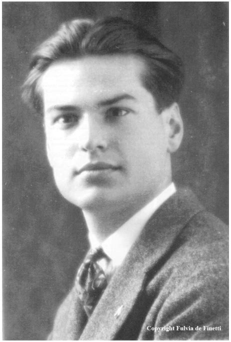
- **Von Neumann-Morgenstern** [3] assumes probabilities are given, derive utilities and expectation from 4 axioms
- **De Finetti** [1] assumes utilities are given, derive probability and expectation from 2 axioms
- **Savage** [2] that assumes neither, derive utilities, probabilities and expectations from 7 qualitative axioms (4 central + 3 more technical)

⇒ one common point in all axiomatics is to assume that **any two acts are comparable** → completeness of preferences

The guys

L. Savage

J. Von Neumann



B. De Finetti

DeFinetti and gambles/acts

- Assume that linear utilities are already given, i.e., an act/gamble

$$a : \mathscr{X} \rightarrow \mathbb{R}$$

  is a function with $f(x)$ the rewards when x happens;
- The price $P(a)$ you consider fair to buy/sell this gamble for reflect your belief ($\sim$ probabilities) about the world
- Give an operational means to measure probabilities

An example

A gamble/ticket *a*, whose reward depends on who win the most sets in next Rolland Garros




	Nadal	Ruud	Cilic	Djokovic
f=	-2	10	0	5

What price $P(a)$ do you associate to this ticket?

Acceptable transaction

The price

$$P(a)$$

is the "fair" price you associate to the ticket/gamble a:

- You would buy for any price $P(a) - \epsilon$, earning

$$a - (P(a) - \epsilon)$$

- You would sell for any price $P(a) + \epsilon$, earning

$$(P(a) + \epsilon) - f$$

→ how should a "rational" agent specify prices?

Transaction on an event

Remember the bet on $A = [1.85, 3]$?

Betting on an event A amounts to play the gamble

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{else} \end{cases}$$

We can use A and $\mathbb{I}_A$ interchangeably, i.e.

$$P(\mathbb{I}_A) = P(A)$$

Avoiding the dutch book[1]

- A set of gambles $a_1, \dots, a_n$
- **You** set prices $P(a_1), \dots, P(a_n)$
- I can sell ($\lambda_i > 0$) or buy ($\lambda_i < 0$) to you any number of gambles
- **You** are **irrational** if there is a dutch book, i.e., a combination with

$$\sup_{x \in \mathscr{X}} \sum \lambda_i \Big(a_i(x) - P(a_i) \Big) < 0,$$

  meaning that whatever happens, you lose money.
- so, a **rational** agent should avoid sure losses when setting prices $P(a_1), \dots, P(a_n)$

[1] History unclear

Probabilities and expectations (exercices)

Do the following:

- Prove that if you are rational, then $\inf a \leq P(a) \leq \sup a$
- Prove that if you are rational, then $P(f+g) = P(f) + P(g)$
- Deduce that $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

A little bit more:

- Show that $\sum_{x \in \mathscr{X}} P(\{x\}) = 1$
- Show that $P(a) = \sum_{x \in \mathscr{X}} a(x) P(\{x\})$

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The first and second properties/axioms are enough to characterize probabilities and expectations.

A simple example

- You bet on raining ($X = R$) or not raining ($X = \neg R$)
- You specify the following prices:

$$P(R) = 0.7, \quad P(\neg R) = 0.5$$

- Show that you can specify numbers $\lambda_R, \lambda_{\neg R}$ of tickets bought/sold such that you are guaranteed to lose.

Additivity of probabilities

- the elements $\mathscr{X} = \{x_1, \dots, x_n\}$
- let $p(x_i)$ be the prices of gamble on x_i
- the sum $\sum_{i=1}^{N} \mathbb{I}_{x_i}(x) = 1$ is constant s(one and only one x_i happens)
- if $\sum_{i=1}^{N} p(x_i) > 1$, buy one ticket each

$$\sum_{i=1}^{N} \mathbb{I}_{x_i}(x) - p(x_i) = 1 - \sum_{i=1}^{N} p(x_i) < 0$$

- if $\sum_{i=1}^{N} p(x_i) < 1$, sell one ticket each

$$\sum_{i=1}^{N} p(x_i) - \mathbb{I}_{x_i}(x) = \sum_{i=1}^{N} p(x_i) - 1 < 0$$

- Hence, to not lose money for sure, **we must have** $\sum_{i=1}^{N} p(x_i) = 1$

Remember our example?

Example (Facing tomorrow's market)

As head of a company selling AI.stuff, you know that the market demand is unlikely to **go crazy** ($\nearrow\nearrow$), will most probably **go up reasonably** ($\nearrow$), but is also unlikely to **remain stable** ($\sim$). You can either:

- Buy a new machine (a_1)
  - If market stable ($\sim$), loss of money (0)
  - If market increase, higher income ($$)
  - If market go crazy, you become rich ($$$)
- Negotiate extra hours with employees if needed (a_2)
  - If market stable ($\sim$), normal income ($)
  - If market increase or go crazy, higher income ($$)
- Do nothing about it (a_3)
  - Whatever happens, normal income ($)

How would you model this decision problem? What would you recommend?

Probability of crazy market?

	$\nearrow\nearrow$	$\nearrow$	$\sim$
a	1000	0	0

- How much your financial analyst would bet on it?
- The more he/she want to bet, the more it is likely
- Since utilities are defined up to affine positive scaling,

$$P(\{\nearrow\nearrow\}) = \mathbb{E}(a)/1000$$

- To see that, consider that

$$\mathbb{E}(a) = 1000 \cdot P(\{\nearrow\nearrow\}) + 0 \cdot P(\{\nearrow\}) + 0 \cdot P(\{\sim\})$$

Probabilities revisited

Main interpretations

- **Frequentist :** $P(A)$= number of times A observed in a population
  - only apply when frequencies make sense
  - what if I want to describe uncertainty on a one-time event?
  - on a quantity that is fixed but ill-known?
- **Subjectivist :** $P(A)$= price for gamble giving 1 if A happens, 0 if not
  - applies to any kind of quantities (even those occuring once or being fixed)
  - e.g.: allows you to attach a probability to the statement "The next united state president will be a woman"

Next lecture

Three options:

- Go see what one of the other guys has to say?

L. Savage

J. Von Neumann

B. De Finetti

- Probabilities and expected utilities are great, cool! But are they perfect? Let's go see some critics of them (and possibly what we can do about it).
- I like probabilities but do not trust utilities, can I do something about it? Let's see some other ways to compare probabilities

References I

[1] B. Finetti.
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[2] L. Savage.
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Wiley, NY, 1954.

[3] J. von Neumann and O. Morgenstern.
Theory of Games and Economic Behavior.
Princeton University Press, 1944.