Decision under uncertainty: Von Neumann and loteries

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Why probabilities?

Can you provide everyday life examples where probabilities are used?

Would you use it in every situations involving uncertainty?

→ This lecture: a justification to the answer "yes", according to Von Neumann

Why utilities, expectation and probability?

In this talk, we will consider

- **Von Neumann-Morgenstern** [5] assumes probabilities are given, derive utilities and expectation from 4 axioms
- **De Finetti** [3] assumes utilities are given, derive probability and expectation from 2 axioms
- **Savage** [4] that assumes neither, derive utilities, probabilities and expectations from 7 qualitative axioms (4 central + 3 more technical)

⇒ one common point in all axiomatics is to assume that **any two acts are comparable** → completeness of preferences

The guys

L. Savage

J. Von Neumann

B. De Finetti

Von-Neumann/Morgenstern and lotteries

- Assume that probabilities over $\mathscr{X}$ (or a suitable Boolean algebra) are given
- Decision makers are then required to compare lotteries

$$L = (\{p(x_1), c_1\}, \dots, \{p(x_n), c_n\})$$

where c_i's are consequences, and x_i's are disjoint events

Axiomatic foundation

We assume an ordering $\succeq$ between lotteries

- Completeness: for two lotteries L_1, L_2, either

$$L_1 \succeq L_2, \; L_1 \preceq L_2$$

- Transitivity: for three lotteries L_1, L_2 and L_3

$$\text{if } L_1 \succeq L_2 \text{ and } L_2 \succeq L_3, \text{ then } L_1 \succeq L_3$$

- Independence of irrelevant alternatives: if L_1, L_2 and L_3 are lotteries with $L_1 \succeq L_2$ and a parameter $t \in [0,1]$, then

$$t\,L_1 + (1-t)\,L_3 \succeq t\,L_2 + (1-t)\,L_3$$

Under these axioms, there is a utility function $u(c)$ on consequences such that

$$L_1 \succeq L_2 \iff \sum_{i=1}^{n} p^1(x_i)u(c_i^1) \geq \sum_{i=1}^{n} p^2(x_i)u(c_i^2)$$

Measuring utilities

Remember that

- There are best and worst consequences $\overline{c}$, $\underline{c}$
- Utilities are defined up to a positive scale, so we can fix

$$u(\underline{c}) = 0 \text{ and } u(\overline{c}) = 1$$

without trouble at any time (before or after measuring)

To measure utility of consequence c, simply compare the lotteries

$$(\{p, u(\underline{c})\}, \{1-p, u(\overline{c})\})$$

and

$$(\{1, u(c)\}).$$

Modify p until they are the same to you.

An example

You want to decide upon your work/leisure balance for this year of study, and the possible consequences are the following:

- c_1: No failed exam, had fun
- c_2: No failed exam, no fun
- c_3: Failed year, had fun
- c_4: Failed year, no fun

How to assess the utility of c_2? What would prefer between

1. No failed exam but no fun for sure, and
2. Having a probability p to fail without fun (worst), and $1 - p$ to succeed with fun (best)?

if

- you prefer 2., then increase p
- you prefer 1., then decrease p
- it is the same to you, then $u(c_2) = p$

An example pursued

Under VNM axiomatic, the statement

Do you prefer “No failed exam but no fun for sure” (c_2) **or**

A 50/50 chance b/w “fail without fun” (c_1) and “succeed with fun” (c_4)

transforms into comparing lotteries

$(\{1, u(c_2)\})$ and $(\{0.5, u(c_1)\}, \{0.5, u(c_4)\})$

If you (strictly) prefer the first, then you state (if you accept the axiomatic)

$$1 \cdot u(c_2) > 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$$

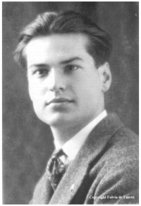
Next lecture

Three options:

- Go see what one of the other guys has to say?

L. Savage

J. Von Neumann



B. De Finetti

- Probabilities and expected utilities are great, cool! But are they perfect? Let's go see some critics of them (and possibly what we can do about it).
- I like probabilities but do not trust utilities, can I do something about it? Let's see some other ways to compare probabilities

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