Decision under uncertainty: decision under ignorance

Sébastien Destercke

CNRS researcher, Laboratoire Heudiasyc, Compiègne

AOS04 master courses

Lecture goal/content

What you will find in this part of the course

- Deciding without any probabilities or with more
  - Deciding without any probability: how?
  - Introduction to more generic framework

Outline

- Decision under risk
  - Basic modelling
  - Complete ranking: one optimal action
  - Incomplete ranking: potentially optimal actions
  - Continuous space of states: intervals

Outline

- Decision under risk
    - Basic modelling
    - Complete ranking: one optimal action
    - Incomplete ranking: potentially optimal actions
    - Continuous space of states: intervals

Classical model

- Each consequence c associated to utility $u(c)$
- Each action/state a/x pair has a utility of the consequence
- Matrix $\mathscr{U}$ becomes $u(a,x)$ values

$$\begin{array}{c|cccc}
 & x_1 & x_2 & \dots & x_n \\
\hline
a_1 & & u(a_1,x_2) & & \\
a_2 & & & & u(a_2,x_n) \\
\vdots & & \ddots & & \\
a_m & & & &
\end{array}$$

An example

We want to cross a sea stretch:

- States: sea weather conditions
- Actions: type of transports

States $\mathscr{X}$

x_1 =Calm sea x_2 =Agitated sea x_3 =Stormy weather



Actions $\mathscr{A}$

a_1 =Motor boat a_2 =Catamaran a_3 =Ferry boat



The matrix $\mathscr{U}$

$$\begin{array}{c|ccc}
 & x_1 & x_2 & x_3 \\
\hline
a_1 & 12 & 0 & -10 \\
a_2 & -2 & 8 & 0 \\
a_3 & 1 & 5 & 10
\end{array}$$

Which action to choose?

What’s the difference with last lecture?

- Last lectures were about using and justifying probabilities
- Here, we assume that only the pay-off matrix is known
- We have no information, whatsoever, about which state obtains
- The only thing we know is what states are **possible**, and what we get when being in one of them

Outline

- Decision under risk
  - Basic modelling
  - Complete ranking: one optimal action
  - Incomplete ranking: potentially optimal actions
  - Continuous space of states: intervals

Maximin: pessimistic behaviour

- For each action a_i, compute $u_\star(a_i) = \min_j u(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $u_\star(a_k) > u_\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$
a_1	12	0	-10	-10
a_2	-2	8	0	-2
a_3	1	5	10	1
Max				①

- We get $a_3 \succ a_2 \succ a_1$, hence a_3 is recommended
- Pessimistic attitude: best action in the worst case

Maximax: optimistic behaviour

- For each action a_i, compute $u^\star(a_i) = \max_j u(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $u^\star(a_k) > u^\star(a_\ell)$

	x_1	x_2	x_3	$u^\star(a_i)$
a_1	12	0	−10	12
a_2	−2	8	0	8
a_3	1	5	10	10
Max				12

- We get $a_1 \succ a_3 \succ a_2$, hence a_1 is recommended
- Optimistic attitude: best action in the best case

In-between: Hurwicz

- Pick a value $\alpha \in [0,1]$, called optimism index
- For a_i, compute

$$u_{H(\alpha)}(a_i) = \alpha u^{\star}(a_i) + (1-\alpha) u_{\star}(a_k)$$

- Say that $a_k >_{\alpha} a_{\ell}$ if $u_{H(\alpha)}(a_k) > u_{H(\alpha)}(a_{\ell})$

	x_1	x_2	x_3	$u_{\star}(a_i)$	$u^{\star}(a_i)$	$u_{H(0.5)}(a_i)$
a_1	12	0	-10	-10	12	1
a_2	-2	8	0	-2	8	3
a_3	1	5	10	1	10	5.5
Max						5.5

- We get $a_3 > a_2 > a_1$, hence a_3 is recommended
- Try to balance between optimistic and pessimistic

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0			
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8		
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	20
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

Savage Minimax regret

- For action a_i, compute $R(a_i,x_j) = \max_k u(a_k,x_j) - u(a_i,x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i,x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	20
a_2	-2	8	0	
$R(a_2)$	14	0	10	14
a_3	1	5	10	
$R(a_3)$	11	3	0	11
Min				11

- We get $a_3 \succ a_2 \succ a_1$, hence a_3 is recommended
- Minimize regret, but sensitive to addition of non-optimal alternatives

Minimax regret vs maximin

Consider the following case:

	x_1	$\cdots$	x_{99}	x_{100}	$R^{\star}(a_i)$
a_1	10	$\cdots$	10	1	
$R(a_1)$					
a_2	2	$\cdots$	2	2	
$R(a_2)$					
Min					

Minimax regret and irrelevant alternatives

Before: $a_3 \succ a_2 \succ a_1$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	−10	
$R(a_1)$				
a_2	−2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
a_4	−5	20	−20	
$R(a_4)$				
Min				

After a_4:

Complete ordering: summary

- Minimax=pessimistic [3]
- Maximax=optimistic
- Hurwicz=in-between [1]
- Savage=Minimizing felt regret [2]

Whatever the chosen rule, we always get one optimal action. But we need to commit to a peculiar behaviour.

What if DM does not want to commit to peculiar behaviour?

What if DM wants to only know the actions that are potentially optimal, given our uncertainty?

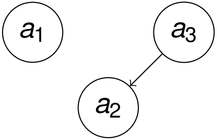
Outline

- Decision under risk
  - Basic modelling
  - Complete ranking: one optimal action
  - Incomplete ranking: potentially optimal actions
  - Continuous space of states: intervals

Lattice ordering

- Say that $a_k \succ a_\ell$ if $u^\star(a_k) > u^\star(a_\ell)$ and $u_\star(a_k) > u_\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10



- Only existing dominance is a_2 by a_3, hence only a_2 is considered non-optimal
- Can be seen as a robust Hurwicz (considering all α as possibilities)
- Note that with this criterion, we eliminate the best action in state x_2

Lattice ordering and information monotonicity

	x_1	x_2	x_3	x_4	$u_\star(a_i)$	$u^\star(a_i)$
a	10	12	14	15	10	15
b	13	11	16	14	11	16

$$b \succ a$$

All states possible

Lattice ordering and information monotonicity

	~~x_1~~	x_2	x_3	x_4	$u_\star(a_i)$	$u^\star(a_i)$
a	~~10~~	12	14	15	12	15
b	~~13~~	11	16	14	11	16

$$b >< a$$

We learn (gain info) x_1 impossible

a and b becomes incomparable.

Lattice ordering and information monotonicity

	~~x_1~~	x_2	~~x_3~~	x_4	$u_\star(a_i)$	$u^\star(a_i)$
a	~~10~~	12	~~14~~	15	12	15
b	~~13~~	11	~~16~~	14	11	14

$$b \prec a$$

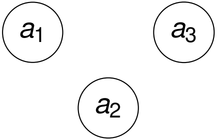
We learn (gain info) x_3 impossible

a is now preferred to *b*.

Interval dominance

- Say that $a_k > a_\ell$ if $u_\star(a_k) > u^\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10

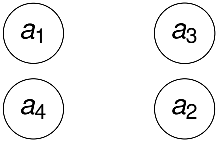


- no dominance at all
- overcautious criterion → may retain Pareto-dominated solutions

Interval dominance: drawback example

- We add a fourth possible, expensive action a_4=Helicopter

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10
a_4	8	8	4	4	8

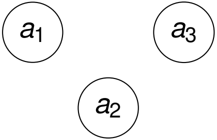


- no dominance at all, even if a_4 better (sometimes strictly) than a_2 in **every** situation!

Difference dominance

- Say that $a_k \succeq a_\ell$ if $u(a_k, x_j) - u(a_\ell, x_j) \geq 0$ for all x_j ($\succ$ if > 0 for at least one x_j)

	x_1	x_2	x_3
a_1	12	0	-10
a_2	-2	8	0
a_3	1	5	10
$a_2 - a_1$	-14	8	10

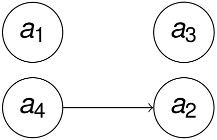


- no dominance at all, again
- do we have the same problem as with interval dominance?

Difference comparison

- We add a fourth possible, expensive action a_4=Helicopter

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10
a_4	8	8	4	4	8
$a_4 - a_2$	10	0	4		



So far...

Options when true state of the world completely unknown:

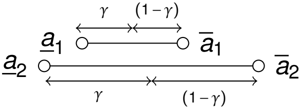
- Complete ordering/one top recommendation
  - Maximin: pessimistic DM
  - Maximax: optimistic DM
  - Hurwicz: attempt to in-between
- Partial ordering/multiple recommendations reflecitng lack of knowledge
  - Lattice ordering: robust hurwicz, may miss potentially optimal actions
  - Interval dominance: very conservative, may keep Pareto dominated options
  - Difference dominance: will keep every non-Pareto dominated solution

Outline

- Decision under risk
  - Basic modelling
  - Complete ranking: one optimal action
  - Incomplete ranking: potentially optimal actions
  - Continuous space of states: intervals

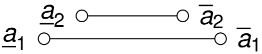
Previous rules revisited

- Alternative a value is represented by an interval $[\underline{u}, \overline{u}]$
- Comparing two alternatives $a_i, a_j \rightarrow$ comparing bounds



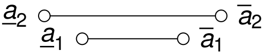
$a_1 \leq_{H(\gamma)} a_2$

Hurwicz ordering



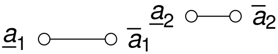
$a_1 \leq_{Mm} a_2$

Maximin



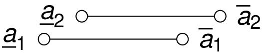
$a_1 \leq_{MM} a_2$

Maximax



$a_1 \leq_{ID} a_2$

Interval dominance



$a_1 \leq_{Lo} a_2$

Lattice ordering

An example

Assume that the evaluation a_i of an object depends on 3 evaluations

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$. Assume we have

	a_i^1	a_i^2	a_i^3	$\min f(a_i^j) = \underline{a}_i$	$\max f(a_i^j) = \overline{a}_i$
a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

An example

Assume that the evaluation a_i of an object depends on 3 evaluations

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$. Assume we have

	a_i^1	a_i^2	a_i^3	$\min f(a_i^j) = \underline{a}_i$	$\max f(a_i^j) = \overline{a}_i$
a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

The decisions

	a_i^1	a_i^2	a_i^3	$\min f(a_i^j) = \underline{a}_i$	$\max f(a_i^j) = \overline{a}_i$
a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

- Maximin is a_3
- Maximax is a_4
- Lattice ordering discards a_1 and a_2
- Interval dominance discards nothing

Difference

Having

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$.

We say that $a_k \succ a_\ell$ under difference dominance if

$$\min_{x,y,z} f(a_k^j) - f(a_\ell^j) \geq 0.$$

Here is an example:

	a_i^1	a_i^2	a_i^3
a_2	6	9	1
a_3	8	8	1

We have $f(a_3) - f(a_2) = x \cdot (8-6) + y \cdot (8-9) + z \cdot (1-1) = 2x - y$

Difference

Having

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$.

We say that $a_k \succ a_\ell$ under difference dominance if

$$\min_{x,y,z} f(a_k^j) - f(a_\ell^j) \geq 0.$$

Here is an example:

	a_i^1	a_i^2	a_i^3
a_2	6	9	1
a_3	8	8	1

We have $f(a_3) - f(a_2) = x \cdot (8-6) + y \cdot (8-9) + z \cdot (1-1) = 2x - y$

$\Rightarrow \min_{x,y,z} f(a_3) - f(a_2) = 4 - 2 = 2$, hence $a_3 \succ a_2$

In our next episode

- Option 1:
  - Probabilities and expected utilities: need a number for every state, actions evaluate by a precise number
  - Sets and bounds: only reckon what is possible (yes/no question), actions evaluated by two bounds

  Let us see how we can reconcile them in a unique framework! → Imprecise probabilities
- Option 2: remember the Ellsberg/Allais paradox? See how we can solve them by still requiring precise evaluations → Rank Dependent Utility

References I

[1] K. J. Arrow and L. Hurwicz.
An optimality criterion for decision-making under ignorance.
Uncertainty and expectations in economics, pages 1–11, 1972.

[2] L. J. Savage.
The theory of statistical decision.
Journal of the American Statistical association, 46(253):55–67, 1951.

[3] A. Wald.
Statistical decision functions.
1950.