Decision under uncertainty: Rank Dependent Utility

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AOS04 master courses

Lecture goal/content

What you will find in this course

- Deciding with probabilities: motivating expected utility
- Deciding without any probabilities + alternatives
    - Deciding without any probability: how?
    - Introduction to more generic framework
    - Alternative ways to decide
- Illustration on classification/learning problem

Outline

- Decision under risk
    - Basic modelling
    - Complete ranking: one optimal action
    - Incomplete ranking: potentially optimal actions
    - Continuous space of states: intervals
- Beyond probabilities and sets
- Other approaches (briefly)

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Classical model

- Each consequence c associated to utility $u(c)$
- Each action/state a/x pair has a utility of the consequence
- Matrix $\mathscr{U}$ becomes $u(a,x)$ values

$$
\begin{array}{c|cccc}
 & x_1 & x_2 & \dots & x_n \\
\hline
 & & & & \\
a_1 & & u(a_1,x_2) & & \\
a_2 & & & & u(a_2,x_n) \\
\vdots & & \ddots & & \\
a_m & & & &
\end{array}
$$

An example

We want to cross a sea stretch:

- States: sea weather conditions
- Actions: type of transports

States $\mathscr{X}$

x_1 =Calm sea x_2 =Agitated sea x_3 =Stormy weather

Actions $\mathscr{A}$

a_1 =Motor boat a_2 =Catamaran a_3 =Ferry boat

The matrix $\mathscr{U}$

$$\begin{array}{c|ccc}
 & x_1 & x_2 & x_3 \\
\hline
a_1 & 12 & 0 & -10 \\
a_2 & -2 & 8 & 0 \\
a_3 & 1 & 5 & 10
\end{array}$$

Which action to choose?

What’s the difference with last lecture?

- Last lecture was about using and justifying probabilities
- Here, we assume that only the pay-off matrix is known
- We have no information, whatsoever, about which state obtains
- The only thing we know is what states are **possible**, and what we get when being in one of them

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Maximin: pessimistic behaviour

- For each action a_i, compute $u_\star(a_i) = \min_j u(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $u_\star(a_k) > u_\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$
a_1	12	0	-10	-10
a_2	-2	8	0	-2
$\textcircled{$a_3$}$	1	5	10	1
Max				①

- We get $a_3 \succ a_2 \succ a_1$, hence a_3 is recommended
- Pessimistic attitude: best action in the worst case

Maximax: optimistic behaviour

- For each action a_i, compute $u^\star(a_i) = \max_j u(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $u^\star(a_k) > u^\star(a_\ell)$

	x_1	x_2	x_3	$u^\star(a_i)$
a_1	12	0	-10	12
a_2	-2	8	0	8
a_3	1	5	10	10
Max				12

- We get $a_1 \succ a_3 \succ a_2$, hence a_1 is recommended
- Optimistic attitude: best action in the best case

In-between: Hurwicz

- Pick a value $\alpha \in [0,1]$, called optimism index
- For a_i, compute

$$u_{H(\alpha)}(a_i) = \alpha u^{\star}(a_i) + (1-\alpha) u_{\star}(a_k)$$

- Say that $a_k >_{\alpha} a_{\ell}$ if $u_{H(\alpha)}(a_k) > u_{H(\alpha)}(a_{\ell})$

	x_1	x_2	x_3	$u_{\star}(a_i)$	$u^{\star}(a_i)$	$u_{H(0.5)}(a_i)$
a_1	12	0	-10	-10	12	1
a_2	-2	8	0	-2	8	3
a_3	1	5	10	1	10	5.5
Max						5.5

- We get $a_3 > a_2 > a_1$, hence a_3 is recommended
- Try to balance between optimistic and pessimistic

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0			
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

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- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	−10	
$R(a_1)$	0	8		
a_2	−2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

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	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

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	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	20
a_2	-2	8	0	
$R(a_2)$				
a_3	1	5	10	
$R(a_3)$				
Min				

Savage Minimax regret

- For action a_i, compute $R(a_i, x_j) = \max_k u(a_k, x_j) - u(a_i, x_j)$ the regret of picking a_i in x_j, instead of the best possible action
- For a_i, compute $R^\star(a_i) = \max_j R(a_i, x_j)$
- Say that $a_k \succ a_\ell$ if $R^\star(a_\ell) > R^\star(a_k)$

	x_1	x_2	x_3	$R^\star(a_i)$
a_1	12	0	-10	
$R(a_1)$	0	8	20	20
a_2	-2	8	0	
$R(a_2)$	14	0	10	14
a_3	1	5	10	
$R(a_3)$	11	3	0	11
Min				11

- We get $a_3 \succ a_2 \succ a_1$, hence a_3 is recommended
- Minimize regret, but sensitive to addition of non-optimal alternatives

Complete ordering: summary

- Minimax=pessimistic [3]
- Maximax=optimistic
- Hurwicz=in-between [1]
- Savage=Minimizing felt regret [2]

Whatever the chosen rule, we always get one optimal action. But we need to commit to a peculiar behaviour.

What if DM does not want to commit to peculiar behaviour?

What if DM wants to only know the actions that are potentially optimal, given our uncertainty?

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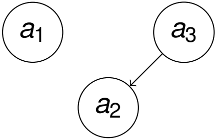
Lattice ordering

- Say that $a_k \succ a_\ell$ if $u^\star(a_k) > u^\star(a_\ell)$ and $u_\star(a_k) > u_\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10

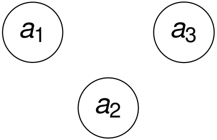


- Only existing dominance is a_2 by a_3, hence only a_2 is considered non-optimal
- Can be seen as a robust Hurwicz (considering all α as possibilities)
- Note that with this criterion, we eliminate the best action in state x_2

Interval dominance

- Say that $a_k > a_\ell$ if $u_\star(a_k) > u^\star(a_\ell)$

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10

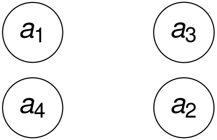


- no dominance at all
- overcautious criterion → may retain Pareto-dominated solutions

Interval dominance: drawback example

- We add a fourth possible, expensive action a_4=Helicopter

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10
a_4	8	8	4	4	8

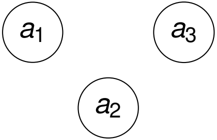


- no dominance at all, even if a_4 better (sometimes strictly) than a_2 in **every** situation!

Difference dominance

- Say that $a_k \geq a_\ell$ if $u(a_k, x_j) - u(a_\ell, x_j) \geq 0$ for all x_j ($\succ$ if > 0 for at least one x_j)

	x_1	x_2	x_3
a_1	12	0	-10
a_2	-2	8	0
a_3	1	5	10
$a_2 - a_1$	-14	8	10

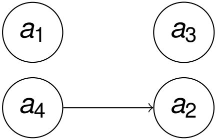


- no dominance at all, again
- do we have the same problem as with interval dominance?

Difference comparison

- We add a fourth possible, expensive action a_4=Helicopter

	x_1	x_2	x_3	$u_\star(a_i)$	$u^\star(a_i)$
a_1	12	0	-10	-10	12
a_2	-2	8	0	-2	8
a_3	1	5	10	1	10
a_4	8	8	4	4	8
$a_4 - a_2$	10	0	4		



So far...

Options when true state of the world completely unknown:

- Complete ordering/one top recommendation
  - Maximin: pessimistic DM
  - Maximax: optimistic DM
  - Hurwicz: attempt to in-between
- Partial ordering/multiple recommendations reflecitng lack of knowledge
  - Lattice ordering: robust hurwicz, may miss potentially optimal actions
  - Interval dominance: very conservative, may keep Pareto dominated options
  - Difference dominance: will keep every non-Pareto dominated solution

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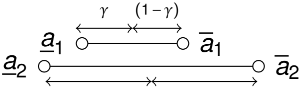
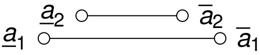
Previous rules revisited

- Alternative a value is represented by an interval $[\underline{u}, \overline{u}]$
- Comparing two alternatives a_i, a_j → comparing bounds



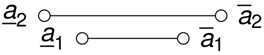
$a_1 \leq_{H(\gamma)} a_2$

Hurwicz ordering



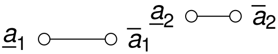
$a_1 \leq_{Mm} a_2$

Maximin



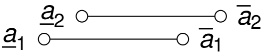
$a_1 \leq_{MM} a_2$

Maximax



$a_1 \leq_{ID} a_2$

Interval dominance



$a_1 \leq_{Lo} a_2$

Lattice ordering

An example

Assume that the evaluation a_i of an object depends on 3 evaluations

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$. Assume we have

	a_i^1	a_i^2	a_i^3	$\min f(a_i^j) = \underline{a}_i$	$\max f(a_i^j) = \overline{a}_i$
a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

An example

Assume that the evaluation a_i of an object depends on 3 evaluations

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

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a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

The decisions

	a_i^1	a_i^2	a_i^3	$\min f(a_i^j) = \underline{a}_i$	$\max f(a_i^j) = \overline{a}_i$
a_1	2	10	3	11	29
a_2	6	9	1	20	37
a_3	8	8	1	23	41
a_4	8	8	9	15	49

- Maximin is a_3
- Maximax is a_4
- Lattice ordering discards a_1 and a_2
- Interval dominance discards nothing

Difference

Having

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$.

We say that $a_k \succ a_\ell$ under difference dominance if

$$\min_{x,y,z} f(a_k^j) - f(a_\ell^j) \geq 0.$$

Here is an example:

	a_i^1	a_i^2	a_i^3
a_2	6	9	1
a_3	8	8	1

We have $f(a_3) - f(a_2) = x \cdot (8-6) + y \cdot (8-9) + z \cdot (1-1) = 2x - y$

Difference

Having

$$a_i = f(a_i^j) = x \cdot a_i^1 + y \cdot a_i^2 + z \cdot a_i^3$$

with each $a_i^j \in [0,10]$ and $x \in [2,3], y \in [1,2], z \in [-1,1]$.

We say that $a_k \succ a_\ell$ under difference dominance if

$$\min_{x,y,z} f(a_k^j) - f(a_\ell^j) \geq 0.$$

Here is an example:

	a_i^1	a_i^2	a_i^3
a_2	6	9	1
a_3	8	8	1

We have $f(a_3) - f(a_2) = x \cdot (8-6) + y \cdot (8-9) + z \cdot (1-1) = 2x - y$

$\Rightarrow \min_{x,y,z} f(a_3) - f(a_2) = 4 - 2 = 2$, hence $a_3 \succ a_2$

A summary so far

Two situations/models:

Sets/decision under risk

- Total ignorance about the true state
- Allow for unique decision (provided DM attitude) or cautious decisions
- May not integrate all our available information

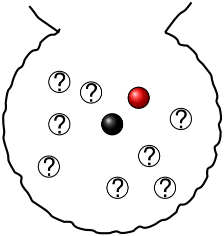
Probabilities

- Requires making precise valued statement about states
- Mostly recommend unique decisions (all axiomatics assume completeness)
- Quite demanding in terms of information

But we can see them as two sides of the same coin!

But first... a game

A bag with ten 🔴 or ● balls, with **at least** one 🔴 and one ●



Now, let's play!

A ticket (action) *a* is such that

- If the drawn ball is 🔴, the ticket *a* owner gets 100 euros
- If the drawn ball is ●, the ticket is worthless

The urn game: interpretation

Prices and gambles

Gamble is a function $a : \mathscr{X} \to \mathbb{R}$ (Urn: $\mathscr{X} = \{$🔴, ⚫$\}$; $a($🔴$) = 100$, $a($⚫$) = 0$)

- $\underline{\mathbb{E}}(a)$ = maximum buying price of a for agent
- $\overline{\mathbb{E}}(a)$ = minimal selling price of a for agent

Buy for any price below $\underline{\mathbb{E}}(a)$, sell for any above $\overline{\mathbb{E}}(a)$, else hold.

Transforming bets in knowledge

- Assumption: decisions reflect our beliefs about world state
- In example: bet acceptance/reject=belief about bag composition

Events as a special case

Event $A \equiv$ function $a \in \{0,1\}$ with $a(x) = 1$ iff $x \in A$

- $\underline{\mathbb{E}}(A)$ = lower probability
- $\overline{\mathbb{E}}(A)$ = upper probability

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Probabilities

Probability mass on finite space $\mathscr{X} = \{x_1, \ldots, x_n\}$ equivalent to a n dimensional vector

$$p := (p(x_1), \ldots, p(x_n))$$

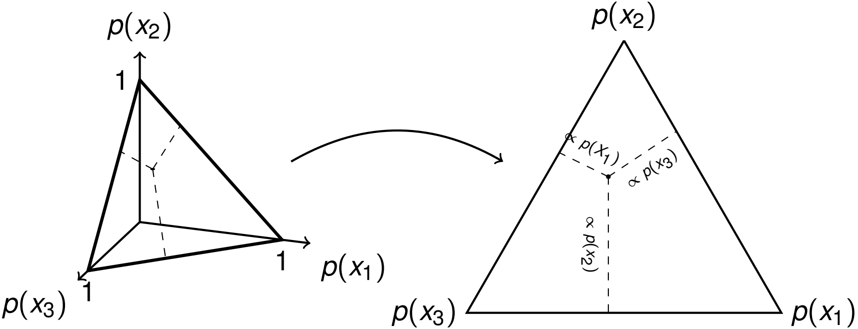
Limited to the set $\mathbb{P}_{\mathscr{X}}$ of all probabilities

$$p(x) > 0, \quad \sum_{x \in \mathscr{X}} p(x) = 1 \quad \text{and}$$

The set $\mathbb{P}_{\mathscr{X}}$ is the $(n-1)$-unit simplex.

Point in unit simplex

$$p(x_1) = 0.2,\ p(x_2) = 0.5,\ p(x_3) = 0.3$$



Imprecise probability

Set $\mathscr{P}$ defined as a set of n constraints

$$\underline{\mathbb{E}}(a_i) \leq \sum_{x \in \mathscr{X}} a_i(x)p(x) \leq \overline{\mathbb{E}}(a_i)$$

where $a_i : \rightarrow \mathbb{R}$ bounded functions

Example

$$p(x_2) - 2p(x_3) \geq 0$$

$$a(x_1) = 0, a(x_2) = 1, a(x_3) = -2, \underline{\mathbb{E}}(a) = 0$$

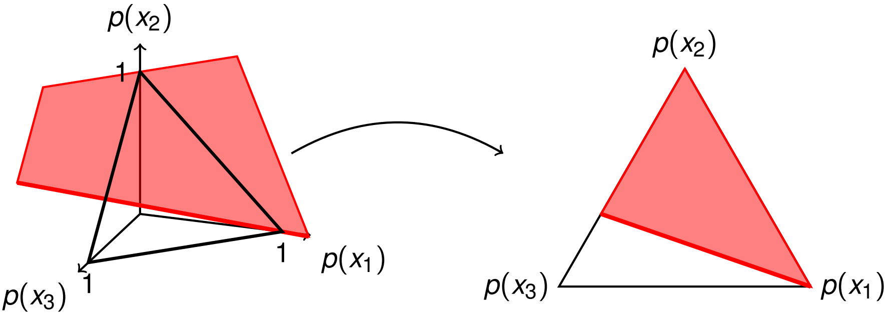
Lower/upper probabilities

Bounds $\underline{P}(A), \overline{P}(A)$ on event A equivalent to

$$\underline{P}(A) \leq \sum_{x \in A} p(x) \leq \overline{P}(A)$$

Set $\mathscr{P}$ example

$$p(x_2) \geq 2p(x_3) \Rightarrow p(x_2) - 2p(x_3) \geq 0$$

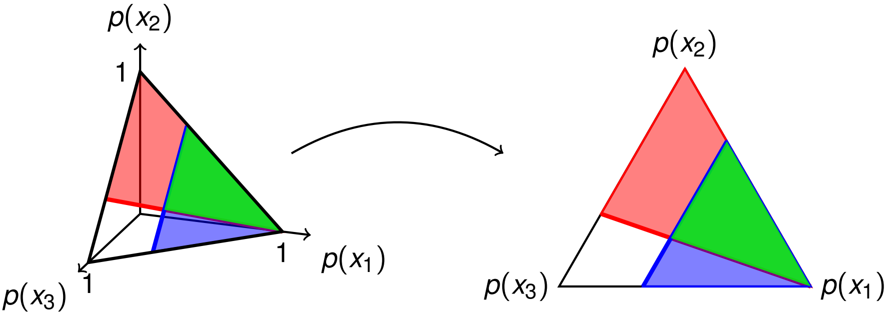


Credal set example

$$p(x_2) - 2p(x_3) \geq 0$$

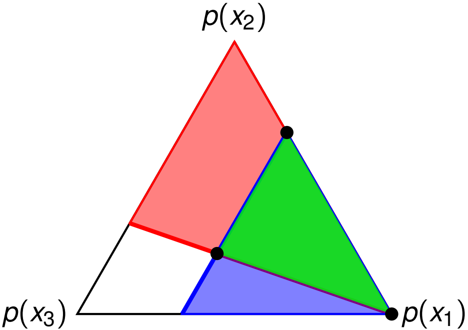
$$p(x_1) \geq {}^1\!/_3$$

$$\mathscr{P}$$



Usual alternative presentation: extreme points

- $p(x_1) = 1, p(x_2) = 0, p(x_3) = 0$
- $p(x_1) = 1/3, p(x_2) = 2/3, p(x_3) = 0$
- $p(x_1) = 1/3, p(x_2) = 4/9, p(x_3) = 2/9$



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Extending expectation

- With one probability p and one act $a : \mathscr{X} \to \mathbb{R}$, we have

$$\mathbb{E}(a) = \sum_x p(x)a(x)$$

- If we consider a (convex) set $\mathscr{P}$ of p's, then we can compute lower/upper bounds

$$\underline{\mathbb{E}}(a) = \inf_{p \in \mathscr{P}} \sum_x p(x)a(x)$$

$$\overline{\mathbb{E}}(a) = \sup_{p \in \mathscr{P}} \sum_x p(x)a(x)$$

which can usually be done by linear programming (easy-peasy).

The basic idea

General case

- Define $\underline{P}, \overline{P}$, and the set $\mathscr{P}$ of probabilities bounded by it
- Expectation $\mathbb{E}$ of an action a becomes interval-valued: $[\underline{\mathbb{E}}(a), \overline{\mathbb{E}}(a)]$
- Apply set-valued ideas to lower/upper expectations

Set case

- we just know $0 \geq p(x_i) \geq 1$ for any x_i
- we retrieve the decision under uncertainty

Probability case

- interval $[\underline{\mathbb{E}}(a), \overline{\mathbb{E}}(a)]$ just reduce to $\mathbb{E}(a) = \underline{\mathbb{E}}(a) = \overline{\mathbb{E}}(a)$
- we retrieve the decision under risk

Decision rules: quick reminder

Total order

- Maximax: compare $\overline{\mathbb{E}}(a)$, take the best!
- Maximin: compare $\underline{\mathbb{E}}(a)$, take the best!
- Hurwicz: weighted average b/w $\overline{\mathbb{E}}(a)$ and $\underline{\mathbb{E}}(a)$

Partial order

- Interval dominance: $a \succ b$ if $\overline{\mathbb{E}}(b) \leq \underline{\mathbb{E}}(a)$
- Lattice: $a \succ b$ if $\overline{\mathbb{E}}(b) \leq \overline{\mathbb{E}}(a) \wedge \underline{\mathbb{E}}(b) \leq \underline{\mathbb{E}}(a)$
- Difference: $a \succ b$ if $\underline{\mathbb{E}}(a - b) \geq 0$

Difference dominance

Under knowledge $\mathscr{P}$, action a_k is better than a_ℓ if

$$\underline{\mathbb{E}}(a_k - a_\ell) = \inf_{p \in \mathscr{P}} \mathbb{E}(a_k - a_\ell),$$

that is if in average, we gain something when exchanging a_ℓ for a_k

Special cases

- probabilities ≡ expected utility
- set ≡ difference dominance (filter out Pareto-dominated solutions)

Outline

- Decision under risk

- Beyond probabilities and sets
  - Imprecise probability: basic ideas
  - Decision with IP
  - Illustration

- Other approaches (briefly)

Back to Ellsberg

9 balls, 3 are reds, 6 remaining are either yellow or black

A

R(ed)	B(lack)	Y(ellow)
100 $	0 $	0$

B

R(ed)	B(lack)	Y(ellow)
0 $	100 $	0$

C

R(ed)	B(lack)	Y(ellow)
100 $	0 $	100$

D

R(ed)	B(lack)	Y(ellow)
0 $	100 $	100$

- What are the possible probability values? In terms of bounds over each colour?
- Compute the lower/upper expectations for each act
- What kind of comparison explain the most frequent behaviour $A \geq B$ but $D \geq C$?

Boat example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

	x_1	x_2	x_3	$\underline{\mathbb{E}}(a_i)$	$\overline{\mathbb{E}}(a_i)$
a_1	12	0	-10	-5	6
a_2	-2	8	0		
a_3	1	5	10		
a_4	8	8	4		

$$\underline{\mathbb{E}}(a_1) = 0 \cdot 12 + 0.5 \cdot 0 + 0.5 \cdot -10 = -5$$

$$\overline{\mathbb{E}}(a_1) = 0.5 \cdot 12 + 0.5 \cdot 0 + 0 \cdot -10 = 6$$

Boat example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0 + \sum p(x) = 1$)

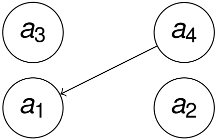
	x_1	x_2	x_3	$\underline{\mathbb{E}}(a_i)$	$\overline{\mathbb{E}}(a_i)$
a_1	12	0	−10	−5	6
a_2	−2	8	0	2	8
a_3	1	5	10	3	7.5
a_4	8	8	4	6	8

- Maximin: a_4
- Maximax: a_4
- Lattice ordering: $a_4 \succ \{a_2, a_3\} \succ a_1$
- Interval dominance: only $a_4 \succ a_1$ (a_2 still possibly optimal)

Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

	x_1	x_2	x_3
a_1	12	0	−10
a_4	8	8	4
$a_4 - a_1$	−4	8	14



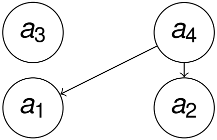
$$\underline{\mathbb{E}}(a_4 - a_1) = 0.5 \cdot -4 + 0.5 \cdot 8 + 0 \cdot -6 = 2$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0 + \sum p(x) = 1$)

	x_1	x_2	x_3
a_2	-2	8	0
a_4	8	8	4
$a_4 - a_2$	6	0	4



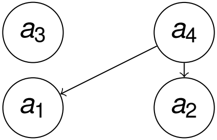
$$\underline{\mathbb{E}}(a_4 - a_2) \geq 0 \text{ because of Pareto-dominance}$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

	x_1	x_2	x_3
a_3	1	5	10
a_4	8	8	4
$a_4 - a_3$	7	3	−6
$a_3 - a_4$	−7	−3	6



$$\underline{\mathbb{E}}(a_4 - a_3) = 0 \cdot 7 + 0.5 \cdot 3 + 0.5 \cdot -6 = -1.5 \text{ and } \underline{\mathbb{E}}(a_3 - a_4) = -5$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

Outline

- Decision under risk
- Beyond probabilities and sets
- Other approaches (briefly)
  - Accounting for risk aversion
  - Alternatives to utilities

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Rank dependent utility (RDU): definition

- act *a* such that $a(x_1) \leq \ldots \leq a(x_n)$
- expected value can be rewritten

$$\sum_{i=1}^{n} p(x_i)a(x_i) = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} p(x_i) \right) (a(x_i) - a(x_{i-1}))$$

  in a cumulative/rank dependent way.
- denoting $r_i = \sum_{j=i}^{n} p(x_i)$ the probability of having at least $a(x_i)$, expected utility

$$\mathbb{E}(a) = \sum_{i=1}^{n} r_i(a(x_i) - a(x_{i-1}))$$

- rank-dependent model suggest to apply an increasing transformation *w* to r_i, with

$$V(a) = \sum_{i=1}^{n} w(r_i)(a(x_i) - a(x_{i-1}))$$

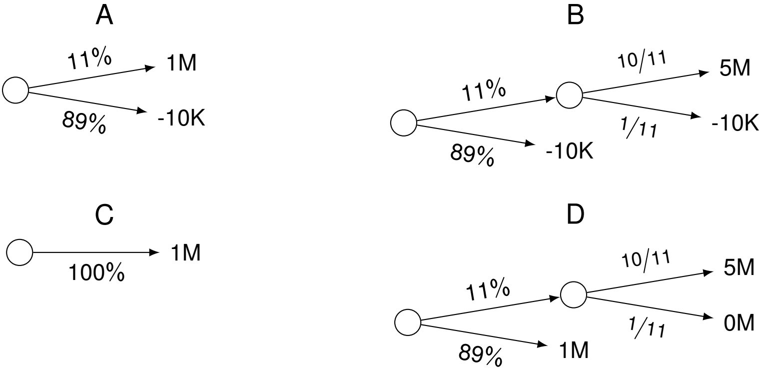
Rank dependent utility (RDU): notes

- If w identity, RDU= expected utility
- If w convex, RDU encodes risk aversion attitude

$$\Rightarrow \text{ solve Ellsberg and Allais paradoxes}$$

- If w concave, RDU encodes risk seeking attitude
- Closely connected to other extensions such as CEU (Choquet Expected Utility)

Revisiting Allais Paradox



- Picking $w(r) = r^2$ explains that most people say $B > A$, but $C > D$
- Same can be done with Ellsberg

Outline

- Decision under risk
- Beyond probabilities and sets
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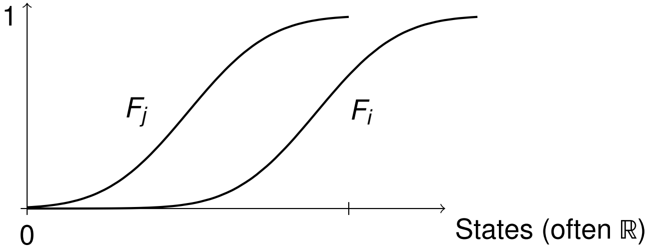
Stochastic dominance

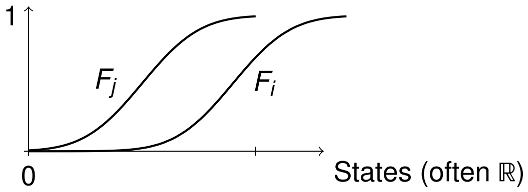
Assumptions and definitions

- States can be arranged from "worst" to "best" linearly
- For action a_i and each state x, we can estimate $F_i(x) = P(]-\infty, x])$

Then a_i stochastically dominates a_j ($a_i \succ_{SD} a_j$) if $F_i(x) \leq F_j(x)$



Why stochastic dominance?



Interests

- $a_i \succ_{SD} a_j \implies \mathbb{E}(a_i) \geq \mathbb{E}(a_j)$ for any increasing utility over $\mathscr{X}$
- Can be seen as a robust version of expected utility
- Only require to be able to rank states (no numbers necessary)

Statistical preference

Assumptions and definitions

- Utilities can be compared with each other (ordered)
- Each action/state is given a (comparable) utility grade

Then a_i is statistically preferred to a_j if $P(a_i \geq a_j) \geq 0.5$

$$p(x_3) = 0.1, p(x_2) = 0.7, p(x_1) = 0.2$$

	x_1	x_2	x_3
a_i	*high*	*high*	*low*
a_j	*low*	*very high*	*very low*
$\mathbb{1}_{a_i \geq a_j}$	1	0	1

$$P(a_i \geq a_j) = 0.1 + 0.2 = 0.3$$

$$P(a_j \geq a_i) = 0.7 \implies a_j \text{ stat. preferred to } a_j$$

Statistical preference and transitivity

Two players play the following game. Player 1 erases the spots from the faces of three fair dice and writes one number from 1, 2,. . . , 18 to each face. Each of them risks 1 €, chooses one dice, they throw the dice, and the one having the bigger number on top of his dice wins the 2 €. Since Player 1 puts the numbers to the dice it seems fair to let Player 2 choose his dice first. Of course, Player 2 tries to choose the best dice. Despite this, Player 1 wins in the long run.

How can that be?

The dice transitivity problem

Consider the following numbering of the fair Dices A, B, C

- $A = \{1, 3, 4, 15, 16, 17\}$
- $B = \{2, 10, 11, 12, 13, 14\}$
- $C = \{5, 6, 7, 8, 9, 18\}$

Assuming that dices are thrown independently, show that we have

$$P(A > B) > \frac{1}{2}, \quad P(B > C) > \frac{1}{2}, \quad P(C > A) > \frac{1}{2},$$

or, in other words, a cyclic statistical preference.

Schematic Wrap-up

Deciding under uncertainty=model of preference (utilities) + model of uncertainty + decision rule

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