# Decision under uncertainty: Imprecise probabilities

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# Lecture goal/content

## What you will find in this lecture

- Deciding without any probability: how?
- Introduction to more generic framework
- Alternative ways to decide

# A summary so far

Two situations/models:

## Sets/decision under risk

- Total ignorance about the true state
- Allow for unique decision (provided DM attitude) or cautious decisions
- May not integrate all our available information

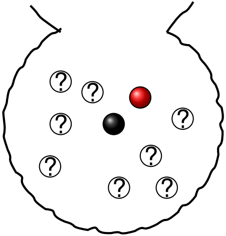
## Probabilities

- Requires making precise valued statement about states
- Mostly recommend unique decisions (all axiomatics assume completeness)
- Quite demanding in terms of information

But we can see them as two sides of the same coin!

## But first... a game

A bag with ten 🔴 or ⚫ balls, with **at least** one 🔴 and one ⚫



### Now, let's play!

A ticket (action) *a* is such that

- If the drawn ball is 🔴, the ticket *a* owner gets 100 euros
- If the drawn ball is ⚫, the ticket is worthless

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# Buy and sell this ticket!

A bag with ten ● (red) or ● (black) balls, with **at least** one ● (red) and one ● (black)

## Now, let's play!

A ticket (action) *a* is such that

- If the drawn ball is ● (red), the ticket *a* owner gets 100 euros
- If the drawn ball is ● (black), the ticket is worthless

For how much would you buy it?

For how much would you sell it?

# The urn game: interpretation

## Prices and gambles

Gamble is a function $a : \mathscr{X} \to \mathbb{R}$ (Urn: $\mathscr{X} = \{●, ●\}$; $a(●) = 100, a(●) = 0$)

- $\underline{\mathbb{E}}(a)$ = maximum buying price of $a$ for agent
- $\overline{\mathbb{E}}(a)$ = minimal selling price of $a$ for agent

Buy for any price below $\underline{\mathbb{E}}(a)$, sell for any above $\overline{\mathbb{E}}(a)$, else hold.

**Transforming bets in knowledge**

- Assumption: decisions reflect our beliefs about world state
- In example: bet acceptance/reject=belief about bag composition

## Events as a special case

Event $A \equiv$ function $a \in \{0, 1\}$ with $a(x) = 1$ iff $x \in A$

- $\underline{\mathbb{E}}(A)$ = lower probability
- $\overline{\mathbb{E}}(A)$ = upper probability

# Coherence revisited [1]

- Assume gambles $a_0, a_1, \ldots, a_n$ and associated prices $\underline{\mathbb{E}}(a_i)$. If $x$ happens, you gain $a_i(x) - \underline{\mathbb{E}}(a_i)$
- Prices are coherent iff

$$\sup_x \left( \sum_{i=1}^{n} (a_i(x) - \underline{\mathbb{E}}(a_i)) - m(a_0(x) - \underline{\mathbb{E}}(a_0)) \right) \geq 0$$

  for $n, m$ non-negative integers
- Prices are coherent iff they are the lower expectations of an associated probability set $\mathscr{P}$ (see later slides)

# Coherence axioms

If gambles $a, b$ live in a linear space $\mathscr{K}$[1], then $\underline{\mathbb{E}}$ is coherent iff

1. $\underline{\mathbb{E}}(a) \geq \inf_x a(x)$
2. $\underline{\mathbb{E}}(\lambda a) = \lambda \underline{\mathbb{E}}(a)$, $\lambda > 0$
3. $\underline{\mathbb{E}}(a+b) \geq \underline{\mathbb{E}}(a) + \underline{\mathbb{E}}(b)$

[1] if $a, b$ in $\mathscr{K}$, then $\lambda a + \kappa b$ in $\mathscr{K}$

# Outline

- Beyond probabilities and sets
  - Imprecise probability: basic ideas
  - Decision with IP
  - Illustration

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# Probabilities

Probability mass on finite space $\mathscr{X} = \{x_1, \dots, x_n\}$ equivalent to a $n$ dimensional vector

$$p := (p(x_1), \dots, p(x_n))$$

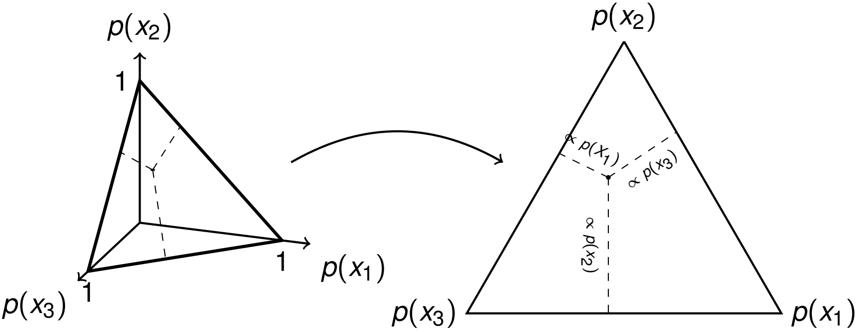
Limited to the set $\mathbb{P}_{\mathscr{X}}$ of all probabilities

$$p(x) > 0, \quad \sum_{x \in \mathscr{X}} p(x) = 1 \quad \text{and}$$

The set $\mathbb{P}_{\mathscr{X}}$ is the $(n-1)$-unit simplex.

# Point in unit simplex

$$p(x_1) = 0.2,\ p(x_2) = 0.5,\ p(x_3) = 0.3$$



# Imprecise probability

Set $\mathscr{P}$ defined as a set of $n$ constraints

$$\underline{\mathbb{E}}(a_i) \leq \sum_{x \in \mathscr{X}} a_i(x)p(x) \leq \overline{\mathbb{E}}(a_i)$$

where $a_i :\to \mathbb{R}$ bounded functions

## Example

$$p(x_2) - 2p(x_3) \geq 0$$

$$a(x_1) = 0, a(x_2) = 1, a(x_3) = -2, \underline{\mathbb{E}}(a) = 0$$

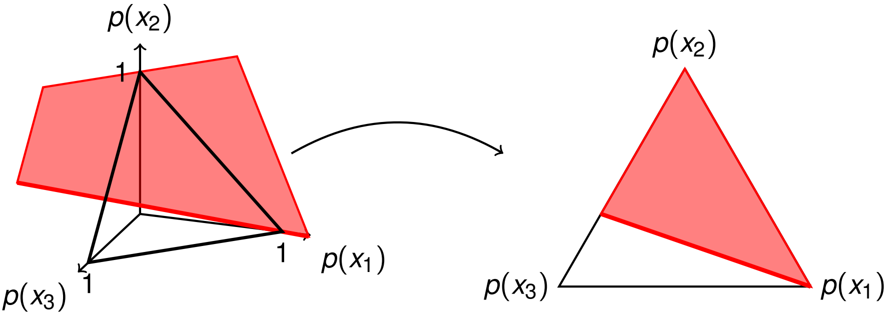
## Lower/upper probabilities

Bounds $\underline{P}(A), \overline{P}(A)$ on event $A$ equivalent to

$$\underline{P}(A) \leq \sum_{x \in A} p(x) \leq \overline{P}(A)$$

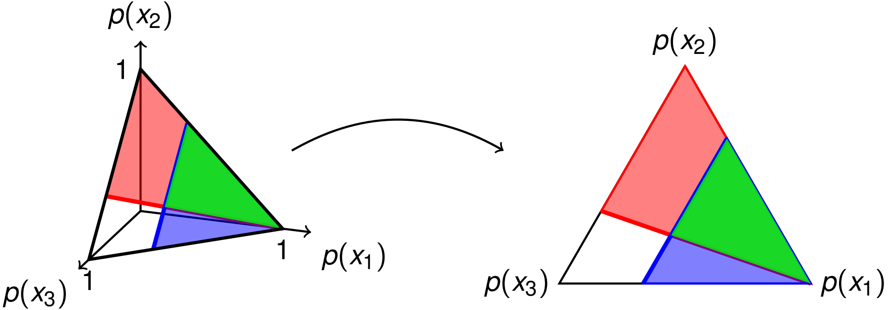
# Set $\mathscr{P}$ example

$$p(x_2) \geq 2p(x_3) \Rightarrow p(x_2) - 2p(x_3) \geq 0$$



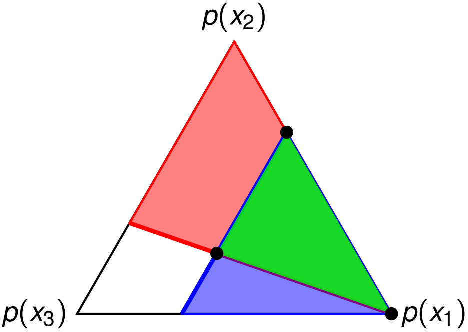
# Credal set example

$$p(x_2) - 2p(x_3) \geq 0$$
$$p(x_1) \geq 1/3$$
$$\mathscr{P}$$



# Usual alternative presentation: extreme points

- $p(x_1) = 1, p(x_2) = 0, p(x_3) = 0$
- $p(x_1) = 1/3, p(x_2) = 2/3, p(x_3) = 0$
- $p(x_1) = 1/3, p(x_2) = 4/9, p(x_3) = 2/9$



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# Extending expectation

- With one probability $p$ and one act $a : \mathscr{X} \to \mathbb{R}$, we have

$$\mathbb{E}(a) = \sum_x p(x)a(x)$$

- If we consider a (convex) set $\mathscr{P}$ of $p$'s, then we can compute lower/upper bounds

$$\underline{\mathbb{E}}(a) = \inf_{p \in \mathscr{P}} \sum_x p(x)a(x)$$

$$\overline{\mathbb{E}}(a) = \sup_{p \in \mathscr{P}} \sum_x p(x)a(x)$$

which can usually be done by linear programming (easy-peasy).

# The basic idea

## General case

- Define $\underline{P}, \overline{P}$, and the set $\mathscr{P}$ of probabilities bounded by it
- Expectation $\mathbb{E}$ of an action $a$ becomes interval-valued: $[\underline{\mathbb{E}}(a), \overline{\mathbb{E}}(a)]$
- Apply set-valued ideas to lower/upper expectations

## Set case

- we just know $0 \geq p(x_i) \geq 1$ for any $x_i$
- we retrieve the decision under uncertainty

## Probability case

- interval $[\underline{\mathbb{E}}(a), \overline{\mathbb{E}}(a)]$ just reduce to $\mathbb{E}(a) = \underline{\mathbb{E}}(a) = \overline{\mathbb{E}}(a)$
- we retrieve the decision under risk

# Decision rules: quick reminder

Total order

- Maximax: compare $\overline{\mathbb{E}}(a)$, take the best!
- Maximin: compare $\underline{\mathbb{E}}(a)$, take the best!
- Hurwicz: weighted average b/w $\overline{\mathbb{E}}(a)$ and $\underline{\mathbb{E}}(a)$

Partial order

- Interval dominance: $a \succ b$ if $\overline{\mathbb{E}}(b) \leq \underline{\mathbb{E}}(a)$
- Lattice: $a \succ b$ if $\overline{\mathbb{E}}(b) \leq \overline{\mathbb{E}}(a) \wedge \underline{\mathbb{E}}(b) \leq \underline{\mathbb{E}}(a)$
- Difference: $a \succ b$ if $\underline{\mathbb{E}}(a - b) \geq 0$

# Difference dominance

Under knowledge $\mathscr{P}$, action $a_k$ is better than $a_\ell$ if

$$\underline{\mathbb{E}}(a_k - a_\ell) = \inf_{p \in \mathscr{P}} \mathbb{E}(a_k - a_\ell),$$

that is if in average, we gain something when exchanging $a_\ell$ for $a_k$

## Special cases

- probabilities ≡ expected utility
- set ≡ difference dominance (filter out Pareto-dominated solutions)

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# Back to Ellsberg

9 balls, 3 are reds, 6 remaining are either yellow or black

A

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 100 $ | 0 $ | 0$ |

B

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 0 $ | 100 $ | 0$ |

C

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 100 $ | 0 $ | 100$ |

D

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 0 $ | 100 $ | 100$ |

- What are the possible probability values? In terms of bounds over each colour?
- Compute the lower/upper expectations for each act
- What kind of comparison explain the most frequent behaviour $A \geq B$ but $D \geq C$?

# Back to Ellsberg

9 balls, 3 are reds, 6 remaining are either yellow or black

A

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 100 $ | 0 $ | 0$ |

B

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 0 $ | 100 $ | 0$ |

C

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 100 $ | 0 $ | 100$ |

D

| R(ed) | B(lack) | Y(ellow) |
|---|---|---|
| 0 $ | 100 $ | 100$ |

# Boat example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$). What is the associated credal set?

# Boat example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

| | $x_1$ | $x_2$ | $x_3$ | $\underline{\mathbb{E}}(a_i)$ | $\overline{\mathbb{E}}(a_i)$ |
|---|---|---|---|---|---|
| $a_1$ | 12 | 0 | −10 | −5 | 6 |
| $a_2$ | −2 | 8 | 0 | | |
| $a_3$ | 1 | 5 | 10 | | |
| $a_4$ | 8 | 8 | 4 | | |

$$\underline{\mathbb{E}}(a_1) = 0 \cdot 12 + 0.5 \cdot 0 + 0.5 \cdot -10 = -5$$

$$\overline{\mathbb{E}}(a_1) = 0.5 \cdot 12 + 0.5 \cdot 0 + 0 \cdot -10 = 6$$

# Boat example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

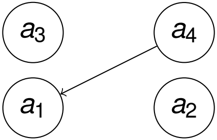
| | $x_1$ | $x_2$ | $x_3$ | $\underline{\mathbb{E}}(a_i)$ | $\overline{\mathbb{E}}(a_i)$ |
|---|---|---|---|---|---|
| $a_1$ | 12 | 0 | −10 | −5 | 6 |
| $a_2$ | −2 | 8 | 0 | 2 | 8 |
| $a_3$ | 1 | 5 | 10 | 3 | 7.5 |
| $a_4$ | 8 | 8 | 4 | 6 | 8 |

- Maximin: $a_4$
- Maximax: $a_4$
- Lattice ordering: $a_4 \succ \{a_2, a_3\} \succ a_1$
- Interval dominance: only $a_4 \succ a_1$ ($a_2$ still possibly optimal)

# Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0 + \sum p(x) = 1$)

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $a_1$ | 12 | 0 | $-10$ |
| $a_4$ | 8 | 8 | 4 |
| $a_4 - a_1$ | $-4$ | 8 | 14 |



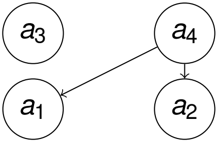

$$\underline{\mathbb{E}}(a_4 - a_1) = 0.5 \cdot -4 + 0.5 \cdot 8 + 0 \cdot -6 = 2$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

# Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0 + \sum p(x) = 1$)

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $a_2$ | $-2$ | 8 | 0 |
| $a_4$ | 8 | 8 | 4 |
| $a_4 - a_2$ | 6 | 0 | 4 |



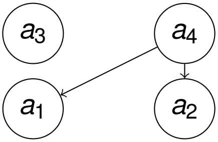

$$\underline{\mathbb{E}}(a_4 - a_2) \geq 0 \text{ because of Pareto-dominance}$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

# Example

Agitated is the most likely state ($p(x_2) \geq p(x_1)$ and $p(x_2) \geq p(x_3)$ + $p(x_i) \geq 0$ + $\sum p(x) = 1$)

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $a_3$ | 1 | 5 | 10 |
| $a_4$ | 8 | 8 | 4 |
| $a_4 - a_3$ | 7 | 3 | −6 |
| $a_3 - a_4$ | −7 | −3 | 6 |




$$\underline{\mathbb{E}}(a_4 - a_3) = 0 \cdot 7 + 0.5 \cdot 3 + 0.5 \cdot -6 = -1.5 \text{ and } \underline{\mathbb{E}}(a_3 - a_4) = -5$$

In the example, difference dominance give $a_4 \succ a_2, a_4 \succ a_1$

# References I

[1] P. Walley.
*Statistical reasoning with imprecise Probabilities*.
Chapman and Hall, New York, 1991.