Université de Technologie de Compiègne
Département de Génie Informatique
Master MS2T, parcours Apprentissage et Optimisation des Systèmes
AOS4 - Décision multi-critères et optimisation multiobjectif
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AOS4 - Multiobjective optimization. Tutorial works

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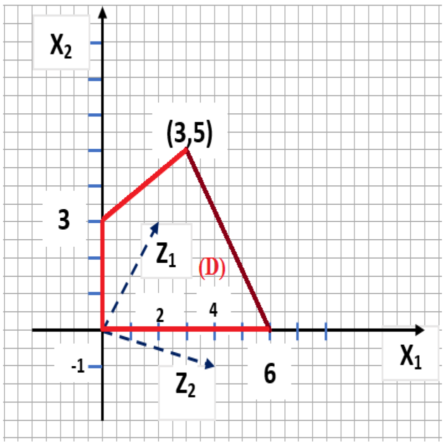
Exercise 1. Consider the following graphics that represents the feasible decision region of a linear two-objective problem:



The problem here is to maximize two linear functions $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ whose gradients are repreented by the two vectors $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ respectively on the graphics.

1. Set up the corresponding multiobjective problem and specify the two objective functions f_i, $i = 1, 2$ and (linear) constraints functions g_j.
2. Graphically represent the feasible objective region $\mathcal{Y}$ in the plane (f_1, f_2).
3. Determiner the payoff table and plot the Nadir point and the ideal point.
4. What is the Pareto set of non-dominated solutions $\mathcal{Y}_N$? Plot the corresponding set of efficient solutions $\mathcal{X}_E$ in the (x_1, x_2) plane.

Exercise 2. Consider the following multi-objective program:

$$\begin{aligned}
&\max \quad f_1 := x_1, \\
&\max \quad f_2 := -5x_1 + 5x_2, \\
&x_1 + x_2 \leq 10, \\
&x_1, x_2 \geq 0.
\end{aligned}$$

In this exercise, we consider the scalarization method of weighted sums and the mono-objective function:

$$f_{eq}(\boldsymbol{x}) = \omega_1 f_1(\boldsymbol{x}) + \omega_2 f_2(\boldsymbol{x}),$$

where $\omega_1, \omega_2 \geq 0$, $\omega_1 + \omega_2 = 1$.

1. Draw the feasible decision region and the feasible objective region.
2. Determine the gradient of the function f_{eq}. With the respective weights $\omega_1 = \frac{3}{4}$ and $\omega_2 = \frac{1}{4}$, determine the respective optimal points.
3. What happens for $\omega_1 = \frac{10}{11}$ and for $\omega_1 = 1$?

Exercise 3. Let $A \in \mathscr{M}_n(\mathbb{R})$ be a symmetric positive definite matrix et let $\boldsymbol{b}_1$, $\boldsymbol{b}_2$ be two vectors of $\mathbb{R}^n$. We will assume that $\boldsymbol{b}_1 \neq \boldsymbol{b}_2$ and $\boldsymbol{b}_1, \boldsymbol{b}_2 \neq 0$ and use the notations $\|\boldsymbol{x}\|_A^2 = \langle A\boldsymbol{x}, \boldsymbol{x}\rangle$.

Consider the two-objective optimization problem

$$(\mathcal{P}) \qquad \min_{\boldsymbol{x}\in\mathbb{R}^n} \begin{cases} f_1(\boldsymbol{x}) = \frac{1}{2}\langle A\boldsymbol{x}, \boldsymbol{x}\rangle - \langle \boldsymbol{b}_1, \boldsymbol{x}\rangle, \\ f_2(\boldsymbol{x}) = \frac{1}{2}\langle A\boldsymbol{x}, \boldsymbol{x}\rangle - \langle \boldsymbol{b}_2, \boldsymbol{x}\rangle, \end{cases}$$

1. Show that each objective function f_i is coercive, i.e.

   $$\lim_{\|\boldsymbol{x}\|\to+\infty} f_i(\boldsymbol{x}) = +\infty,$$

   $i = 1, 2$.
2. What is the solution $\boldsymbol{x}_1^\star$ (resp. $\boldsymbol{x}_2^\star$) of the mono-objective problem

   $$\min_{\boldsymbol{x}} f_1(\boldsymbol{x}) \qquad \Big(\text{resp. } \min_{\boldsymbol{x}} f_2(\boldsymbol{x})\Big) \text{ ?}$$

   In the sequel we will use the notations $f_1^\star = f_1(\boldsymbol{x}_1^\star)$ and $f_2^\star = f_2(\boldsymbol{x}_2^\star)$.
3. Show that

   $$f_1^\star = -\frac{1}{2}\|\boldsymbol{b}_1\|_{A^{-1}}^2$$

   and that for all $\boldsymbol{x} \in \mathbb{R}^n$,

   $$f_1(\boldsymbol{x}) = f_1^\star + \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_1^\star\|_A^2.$$

   Deduct the shape of the level sets for f_1. Do the same for f_2.
4. In the sequel we will denote $\mathcal{Y}$ the set of feasible objectives in the space (f_1, f_2), $\mathcal{Y}_N$ the Pareto set and $\mathcal{X}_E \subset \mathbb{R}^n$ the efficient set. From a geometric arguments, show that the efficient set $\mathcal{X}_E$ is

   $$\mathcal{X}_E = \{\boldsymbol{x}^\star \in \mathbb{R}^n \ / \ \boldsymbol{x}^\star = \theta\boldsymbol{x}_1^\star + (1-\theta)\boldsymbol{x}_2^\star,\ \theta \in [0,1]\}.$$

5. In this question, one tries to derive the general shape of the region of feasible objectives and we will determine the Pareto set $\mathcal{Y}_N$ of problem $(\mathcal{P})$.

   (a) Determine the ideal point I of the problem $(\mathcal{P})$.

   (b) For $\boldsymbol{x}^\star = \theta \boldsymbol{x}_1^\star + (1-\theta)\boldsymbol{x}_2^\star$ $(\boldsymbol{x} \in \mathcal{X}_E)$, show that

$$f_1(\boldsymbol{x}^\star) = f_1^\star + \frac{1}{2}(1-\theta)^2 \|\boldsymbol{x}_2^\star - \boldsymbol{x}_1^\star\|_A^2,$$

$$f_2(\boldsymbol{x}^\star) = f_2^\star + \frac{1}{2}\theta^2 \|\boldsymbol{x}_2^\star - \boldsymbol{x}_1^\star\|_A^2.$$

   For the sake of simplicity, we will denote $K = \frac{1}{2}\|\boldsymbol{x}_2^\star - \boldsymbol{x}_1^\star\|_A^2$. Show that

$$[K - (f_1(\boldsymbol{x}^\star) - f_1^\star) - (f_2(\boldsymbol{x}^\star) - f_2^\star)]^2 = 4(f_1(\boldsymbol{x}^\star) - f_1^\star)(f_2(\boldsymbol{x}^\star) - f_2^\star)$$

   which is the equation of an ellipse.

   (c) What are the end points of $\mathcal{Y}_N$?

   (d) Show that

$$\lim_{\|\boldsymbol{x}\| \to +\infty} \frac{f_1(\boldsymbol{x})}{f_2(\boldsymbol{x})} = 1.$$

6. Let us recall that, in mono-objective optimization theory, a descent direction $\boldsymbol{d}$ for the function f_1 at point $\boldsymbol{x}$ $(\nabla f_1(\boldsymbol{x}) \neq 0)$ is such that

$$\langle \boldsymbol{d}, \nabla f_1(\boldsymbol{x}) \rangle < 0.$$

   Here, at a point $\boldsymbol{x}$, one tries to have a descent direction $\boldsymbol{d}$ for both f_1 and f_2. Let

$$\mathcal{C}_x = \{\boldsymbol{d} \in \mathbb{R}^n,\ \langle \boldsymbol{d}, \nabla f_1(\boldsymbol{x}) \rangle < 0,\ \langle \boldsymbol{d}, \nabla f_2(\boldsymbol{x}) \rangle < 0\}.$$

   When $\mathcal{C}_x$ is non empty, what is the nature of this set ?
   Under which conditions the set $\mathcal{C}_x$ is empty ?