Decision under uncertainty: the case of propositional logic

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AOS04 master courses

Lecture goal/content

What you will find in this course

- Introduction on how to use propositional logic and IP

Propositional logic

- We assume (finite) variables $\{a, b, c, d, \ldots\}$ that can be true or false
- A formula ϕ combines variables and $\vee, \wedge, \neg$
- The set Ω is the set of possible interpretations:
  - An interpretation ω satisfies (or is a model of) ϕ if it makes ϕ true

$$\omega \models \phi$$

  - The set $E_\phi \subseteq \Omega$ of models of ϕ is a subset/an event of Ω, i.e.,

$$E_\phi = \{\omega \in \Omega : \omega \models \phi\}$$

An example

Alice, Bob and Carl will go or not to a party, denoted $\{a, b, c\}$. What does model the next formula, and what is the model subset E_ϕ

$$\phi = (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$$

Probabilities and logic

We will work with a particular assumption:

- We assume that to each variable x is associated either
  - A probability $p(x) = 1 - p(\neg x)$ or
  - An interval probability $[\underline{p}(x), \overline{p}(x)] = [1 - \overline{p}(\neg x), 1 - \underline{p}(\neg x)]$
- We have a formula for which we want to assess the probability of it happening, i.e., what do we know about

$$P(E_{\phi})$$

given what we know about each variable.

A very first problem

Consider two variables $\{a, b\}$ and the following formula:

$$\phi = a \wedge b$$

Assume first that we have

$$p(a) = 0.4, p(b) = 0.5$$

. What is

- The probability of ϕ assuming independence of variables?
- The probability of ϕ assuming we do not know the dependence?

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A very first problem

Consider two variables $\{a, b\}$ and the following formula:

$$\phi = a \wedge b$$

Assume first we have

$$[\underline{p}(a), \overline{p}(a)], [\underline{p}(b), \overline{p}(b)]$$

. What is

- The probability of ϕ assuming independence of variables?
- The probability of ϕ assuming we do not know the dependence?

A generic formulation, assuming independence

Consider ϕ and E_ϕ. For each $\omega \in E_\phi$, denote by

- $\mathcal{N}_\omega$ the set of negated variables in ω,
- $\mathcal{P}_\omega$ the set of positive variables in ω.

Then, under independence, we have

$$P(E_\phi) = \sum_{\omega \in E_\phi} \prod_{x \in \mathcal{P}_\omega} p(x) \prod_{x \in \mathcal{N}_\omega} 1 - p(x).$$

Given $[\underline{p}(x), \overline{p}(x)]$, we would like to compute

$$\underline{P}(E_\phi) = \inf_{p(x) \in [\underline{p}(x), \overline{p}(x)]} P(E_\phi), \quad \overline{P}(E_\phi) = \sup_{p(x) \in [\underline{p}(x), \overline{p}(x)]} P(E_\phi)$$

We can prove that the results are obtained either for $p(x) = \underline{p}(x)$ or $p(x) = \overline{p}(x)$

Back to our example

An example

Alice, Bob and Carl will go or not to a party, denoted $\{a, b, c\}$. What does model the next formula, and what is the model subset E_ϕ

$$\phi = (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$$

What is the probability of it happening if we know:

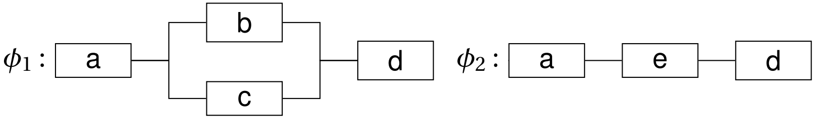
- $p(a) \in [0.9, 1]$
- $p(b) \in [0.5, 0.5]$
- $p(c) \in [0.2, 0.4]$

A more concrete example

We consider five elements $\{a, b, c, d, e\}$ with reliabilities:

- $p(a) \in [0.7, 0.8]$
- $p(b) \in [0.9, 1]$
- $p(c) \in [0.9, 1]$
- $p(d) \in [0.6, 1]$
- $p(e) \in [0.8, 0.9]$

Assume also the two following systems ϕ_1, ϕ_2:



What are the probabilities $[\underline{P}(\phi_i), \overline{P}(\phi_i)]$?
Which system is better? ϕ_1 or ϕ_2?