St. Petersburg Paradox

When you have calculated the expected value in Exercise 2 with an unlimited amount of resources you should have found out that the expected value is infinite.
Considering the expected value, one should play the game at any price if offered the opportunity.
The paradox is the classic situation in which the direct application of decision theory (which takes into account only expected gain) suggests a course of action that no reasonable person would feel comfortable following.
This paradox is very famous and there are lots of "solutions" proposed in time and critics. I propose a brief summary of the origin of this paradox, the main solutions proposed in time and the main critics.

1 St. Petersburg Paradox origin

The St. Petersburg paradox is a thought experiment in probability and decision theory that is named after the journal in which it was first published. It was first described by Daniel Bernoulli in a paper titled "Specimen Theoriae Novae de Mensura Sortis" in 1738. The paradox illustrates the counter-intuitive nature of using expected value to evaluate uncertain outcomes, and has been widely discussed and debated in the fields of economics and decision theory.
The initial version of the game was more complicated (6 dices) by the one we have seen in the exercise, then it has been simplified by G. Cramer.

2 Gabriel Cramer solution

Gabriel Cramer is a Geneve mathematician that had read about Barnoulli's problem in a book of Montmort.
He argued that monetary value is subjective and the notion of expected value should not guide the decision on the bet. In fact, he argued that money start to saturate in value beyond a certain point. For example, twenty million is not worth ten million more than ten million, because the first ten million would be enough for satisfying all the needs and desires of a rational player.

2.1 The Upper Bound Solution

We know that wordly money are finite and, for this reason, Cramer proposed to put an upper bound to the prize money for the game.

St. Petersburg lotteries with upper bound Suppose now that the maximum value that the casino can pay is **W**. In fact, assuming that the casino cannot pay more than **W**, the game statement should predict that after L consecutive draws of Head for which $2^L = W$, the prize **W** is paid, but the draw (L+1) is not continued and the game begins again, starting from 1. In this case, the expected value becomes:
$E = \sum_{k=1}^{L+1} p_k 2^{k-1} = \sum_{k=1}^{L+1} \frac{1}{2} = \frac{L+1}{2}$
In practice, the expected value of the prize is thus proportional to the logarithm in base 2 of the maximum prize. The table below shows the expected premium values as a function of the maximum premium value payable by any bank:

Banker	Max win	Number ° max of heads	Expected value
Children	8	3	2
Game between friends	64	6	3.5
Millionaire	$1'050'000$	20	10.50
Billionaire	$1'075'000'000$	30	15.50
Fantastiliardary	10^{100}	333	166.50

It can be seen that while the maximum premium grows very rapidly, exponentially, the average premium value grows very, very slowly, logarithmically. An average person might still find the game unattractive if he or she were to pay an entry fee comparable to the expected premium shown in the table. However, the discrepancy between intuition and the theoretical calculation of expected value is much less dramatic than in the initial case.

Critics: The assumptions of adding an upper bound limit its shortcomings. Firstly, it is controversial to claim that an upper boundary exists. Any mathematician would argue that just because worldly resources are finite doesn't mean that the expected value should not be allowed to tend to infinity.

The second critics is that we might formulate the St. Petersburg lottery by switching dollars with grains of sand (with are infinite). Cramer's was aware of this and so he proposed another solution.

2.2 The moral value solution

In this solution Cramer proposed a reduced speed of growth of monetary value by proposing the following procedure: he defined the moral value of goods as the square root of the mathematical quantities. According to Cramer, the moral value of an outcome is determined by the degree of satisfaction it brings to the decision-maker, and this satisfaction should be taken into account when calculating the expected value of an uncertain outcome. He indirectly as proposed the **law of diminishing marginal utility**.
In this way the value of additional wealth never reaches zero, but one gains lesser and lesser as he increases his wealth further.

3 Daniel Bernoulli solution

> The determination of the value of an item must not be based on the price, but rather on the utility it yields ... There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.
>
> Daniel Bernoulli

Bernoulli has solved the problem by introducing explicitly the concept of *utility function*, the *expected utility hypothesis* and the *law of diminishing marginal utility.*

3.1 Expected utility hypothesis

The expected utility hypothesis is a popular concept in economics that serves as a reference guide for decisions when the payoff is uncertain.
The expected utility hypothesis states an agent chooses between risky prospects by comparing expected utility values. In this paradox the difficulty is to define the utility model for defining the utility values. Bernoulli has proposed the *log utility* model.

3.2 Diminishing marginal utility

- Utility is the degree of satisfaction or pleasure a consumer gets from an economic act. For example, a consumer can purchase a sandwich so they are no longer hungry, thus the sandwich provides some utility.

- Marginal utility is the enjoyment a consumer gets from each additional unit of consumption. It calculates the utility beyond the first product consumed. If you buy a bottle of water and then a second one, the utility gained from the second bottle of water is the marginal utility.

- The **law of diminishing marginal** utility directly relates to the concept of diminishing prices. As the utility of a product decreases as its consumption increases, consumers are willing to pay smaller dollar amounts for more of the product.

Example:
A person who is hungry may experience a great utility from eating a sandwich, but may feel a smaller marginal utility from eating a second sandwich.

3.3 First Bernoulli solution

A common utility model, suggested by Daniel Bernoulli, is the logarithmic function $U(w) = \ln(w)$ (known as log utility). It is a function of the gambler's total wealth w, and the concept of diminishing marginal utility of money is built into it. The expected utility hypothesis says that the utility function provides a good criterion for real people's behavior; a function that returns a positive or negative value indicating if the wager is a good gamble. For each possible event, the change in utility ln (wealth after the event) $-$ ln (wealth before the event) will be weighted by the probability of that event occurring. Let c be the cost for the lottery's ticket. The expected incremental utility of the lottery now converges to a finite value:

$$\Delta E(U) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \left[\ln\left(w + 2^k - c\right) - \ln(w)\right] < +\infty.$$

This formula gives an implicit relationship between the gambler's wealth and how much he should be willing to pay (specifically, any c that gives a positive change in expected utility). For example, with natural log utility, a millionaire ($\$1,000,000$) should be willing to pay up to \$20.88, a person with $\$1,000$ should pay up to \$10.95, a person with \$2 should borrow \$1.35 and pay up to \$3.35.

Cramer demonstrated that a square root function describing the diminishing marginal benefit of gains can resolve the problem. Bernoulli has used a similar approach but he also took into account the wealth of the person.

3.4 Second Bernoulli solution

Nicolas Bernoulli himself proposed an alternative idea for solving the paradox. He conjectured that people will neglect unlikely events. Since in the St. Petersburg lottery only unlikely events yield the high prizes that lead to an infinite expected value, this could resolve the paradox. But this solution has been criticized for the fact some people behave enjoy the risk.